# A contest success function for networks*

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**Abstract**

This paper models conflict within a network of friendships and enmities between players. We assume that each player is either in a friendly or in an antagonistic relation with every other player and players compete for a fixed prize by exerting costly efforts. We axiomatically characterize a success function which determines the share of each player given the efforts and the network of relations. This framework allows for the study of strategic incentives and friendship formation under conflict as well as the application of stability concepts of network theory to contests.

*Keywords:* contest, conflict, success function, network, alliance, pairwise stability.
*JEL classification:* C70, D72, D74, D85.

## 1 Introduction

In many conflictual situations, we often observe that competing parties join forces to fight together against others or refrain from fighting with each other. For instance, lobby groups may cooperate in supporting the same legislation when their interests coincide; belligerent states may form alliances for joint action if they face a common threat; competing firms may collude to increase their market share and so on. These parties do not necessarily act in a perfectly coordinated way, especially when their relation is an occasional opportunistic cooperation rather than a long term commitment. Such relations usually rely on informal bilateral agreements and may lead to a complex network.

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This paper models conflict as a contest, where players compete for shares of some fixed resources by exerting costly efforts. We assume that each pair of players is either in a friendly or in an antagonistic relation, and the relations between all pairs define a network. In this setting, we propose and axiomatically characterize a *success function* which determines the share of resources of each player given all efforts and the network of relations. So far, the axiomatic work in the contest literature has exclusively focused on conflict between groups (e.g., Münster, 2009) or between individuals (e.g., Skaperdas, 1996). In the former, players are divided into mutually exclusive groups and groups compete with each other, while in the latter each player competes individually against all others. Yet, many competitive situations may lead to networks different than the *all against all* or *groups against groups* types of networks. For instance, in international relations most alliances between states do not mean perfect coordination or long term commitments, and *the friend of a friend can be an enemy* at times. In his categorization of strategic alliances, Ghez (2011) defines the "tactical" alliance as a form of state alignment which occurs when states encounter a common immediate threat. The opportunistic nature of tactical alliances does not exclude a partnership between states which are not members of the same coalition. For instance, the United States and the Soviet Union were well-known to be enemies in the 1980s. However, they both supported Iraq in the Iran-Iraq War as they both perceived post-revolutionary Iran as a threat. We illustrate the corresponding network in Figure 1, which is neither all against all nor groups against groups type.[1] Our paper extends the axiomatic foundations to contests with any type of network of relations. As a starting point, we propose a

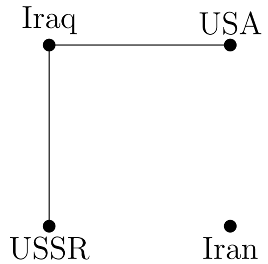


Figure 1: The network describing the relationships among the US, the USSR, Iran and Iraq during Iran-Iraq war.

class of success functions which we derive through a probabilistic argument following well-known results by McFadden (1973). This class is very general and particularly convenient in a variety of applications. Given that the network variable is new and our aim is to understand how shares depend on the relations as well as efforts, we provide an axiomatic characterization. Our characterization consists of six axioms, three of which are direct extensions of the well-known anonymity, monotonicity and exhaustivity axioms in the literature. We define three new axioms; namely, the *independence of efforts of commons*, the *independence of relations of others* and the

[1]For further examples of tactical alliances and definitions of other categories, see Ghez (2011). We discuss an example on political lobbying in Section 5.2.

*monotonicity of relations*, which specify how the network variable affects the behavior of the success function. Once we restrict our attention to all against all contests, our functional form belongs to the well-established class characterized by Skaperdas (1996). For contests between groups, our class does not immediately link to the class axiomatized in Münster (2009) as our function determines shares of individual players rather than groups. However, the function obtained by summing up the shares of group members derived by our function belongs to the class axiomatized in Münster (2009).

Our framework is useful in connecting two major fields, namely contests and network theory. Our model can in fact be used to study a specific network formation problem where the total value of the network is fixed and the success function works as the allocation rule for given efforts of players. We state this problem and show that under symmetric efforts the unique pairwise stable network is the peace network, where all players are friends. In the context of international relations, this result simply means that if countries had equal power, the only stable outcome would be peace. Alternatively, our model can be used to study strategic choice of efforts given a network of relations, with Nash equilibrium as the solution concept. This would be the standard setting in contest theory if the network was all against all. We provide an example where we consider all networks with equal number of friends for each player, and show that aggregate equilibrium efforts decrease in the number of friendships. Since our functional form provides a unified framework to study conflict within any type of network, we hope it will lead to further applications in conflict theory.

An immediate extension of our model is to allow for different degrees of friendships across players. We extend our probabilistic derivation to the case of weighted directed networks, which result from friendships that are not necessarily mutually valued. So, a player can have different degree of friendliness towards every other player and the degree of friendliness between two players is not necessary mutual. The resulting generalized functional form naturally contains the benchmark case of undirected unweighted networks. Our axiomatization of the benchmark case provides a deeper understanding of the generalized functional form as well but does not immediately extend. Since the network variable is new in the axiomatic work on contest success functions, it is worth starting simple and we leave the axiomatization of the generalized class to future research.

Section 2 reviews the literature. We introduce our model formally in Section 3 and provide the probabilistic derivation of a functional form. In Section 4, we characterize this functional form via six axioms and show that our characterization is tight by means of examples of functions which satisfy all axioms but one. In Section 5.1 and Section 5.2, we discuss applications of our analysis to the network formation problems and contest games respectively. We extend our probabilistic derivation to the case of weighted directed networks in Section 5.3. Section 6 concludes. All proofs are in Appendix.

# 2 Related literature

The contest is the workhorse model for representing conflict and competition over scarce resources. Contest models have been applied in a variety of areas of economics and social sciences, such as rent seeking, industrial organization, incentives within organizations and armed conflict. See Konrad (2009) for an introduction to contest theory and its applications.

A crucial element of a contest model is the success function, which generally defines the mapping from individual efforts to shares of resources. The nature of a contest fundamentally depends on the features of this functional form. Foundational work on success functions is divided into two leading approaches; namely, axiomatic and stochastic approaches. The axiomatic approach to contest success functions started with the seminal work by Skaperdas (1996). This characterization has been extended in several directions by relaxing some of the axioms (e.g., Clark and Riis, 1998, Blavatskyy, 2010; see Jia et al., 2013, for a review), by generalizing to multi-dimensional efforts of players (e.g., Rai and Sarin, 2009, Arbatskaya and Mialon, 2010), or by allowing rankings as the outcome of a contest instead of a single winner (Vesperoni, 2014, Lu and Wang, 2015). While these contributions are exclusively on all against all contests, Münster (2009) axiomatically characterizes a success function for contests where the competition takes place between mutually exclusive groups of players, i.e., coalitions. Such networks are very important in several contexts; however, they constitute only a small subset of possible networks in conflict. Our paper follows the axiomatic approach in contests, allowing players to compete in every possible network of friendships and enmities. As for the second body of literature on the foundations of success functions, it is concerned with stochastic derivations of success functions following techniques from discrete choice econometric models. The stochastic approach so far has focused on success functions for all against all contests (Hirshleifer and Riley, 1992, Jia, 2008, Fu and Lu, 2011; see Jia et al., 2013 for a review). Together with the axiomatic characterization, we motivate our function by deriving it following the well-known probabilistic arguments in McFadden (1973).

We now selectively review works which study the broad subject of conflict when players compete in networks or are linked to each other in a way that may indirectly define a network. None of these works is axiomatic, nor their focus is on the success function. A recent paper by König et al. (2015) considers conflict within networks of alliances, where they propose a success function that determines each player's share of the prize. A difficulty with their function is that it might lead to negative shares for some networks and effort profiles. They show that the equilibrium effort of a player is related to an index of its centrality in the network under some restrictions, and they perform an empirical analysis using data from the Second Congo War. This work is an important advance into directions different from our work which is concerned with the axiomatic foundations of the conflict mechanism. A related body of literature is the work on sabotage contests where players direct a specific effort to handicap each particular opponent (e.g., Konrad, 2000, Gürtler, 2008, Gürtler and Münster, 2010; see Amegashie, 2015 for a review). As the efforts are opponent specific, the intensity of competition between each pair of players is different like in our work; however, besides the differences in the theoretical approach (such as multiple efforts, non-axiomatic approach) they do not allow for alliances. Other related work with indirectly defined networks is on contests with identity dependent externalities. In these papers, players typically value victory identically; however, the value of defeat depends on the identity of the winner. The identity-based losses of all pairs can be seen as defining a network of relations between players. Applications of this setup are to model ethnic conflict between minorities (e.g., Esteban and Ray, 1999, 2011) or political lobbying between parties on an ideological spectrum (e.g., Klose and Kovenock, 2012, 2013).

Several papers in the subject of coalition formation in conflict study contests between groups. Following the seminal contribution of Olson (1965), these works focus on the collective action problem in groups, showing that the power of a group may not increase in its size due to free-riding in the provision of collective effort. See Bloch (2010) for a review on endogenous formation of groups in conflict. Other papers consider contests with multiple battlefields; see Kovenock and Roberson (2012b) for an introduction to this literature. In this setting, Kovenock and Roberson (2012a) study incentives for alliance formation in the sense of pooling resource budgets when fighting a common adversary on separate battlefields. They depart from the above mentioned coalition formation literature by assuming that an ally's effort is completely rival rather than a public good. In that sense, their definition of an alliance is in line with our use of the tactical alliances.

Although not strictly related as they focus on network analysis, there are contributions on the broader subject of conflict within networks. Hiller (2011) analyzes a model where there are as many local conflicts as pairs of players and the win probabilities for each pair are determined by the number of their friends. In this model, there is no endogenous choice of efforts and payoffs are fully determined by the network of relations. Franke and Öztürk (2015) consider players embedded in a network of bilateral conflicts and each pair can choose to fight in each conflict by spending efforts or refrain from fighting. In this setting, they characterize equilibrium efforts given specific types of conflict networks. Jackson and Nei (2014) define and analyze a new solution concept, called war-stability, for networks where each player is in a friendly or antagonistic relation with every other player. A necessary condition for war-stability is that no coalition of players can successfully attack another coalition. Unlike us, they associate a fixed effort to each player and their success function is deterministic. Goyal and Vigier (2014) consider a two-player game where a designer chooses a network and allocates specific efforts to defend each node, while an adversary allocates specific efforts to attack each node after observing these. They find the optimal network structure for the designer when the pair of efforts determines the probability of destruction of each node via Tullock (1975) success function.

# 3 Modeling networks in conflict

We consider a set of players $N = \{1, \ldots, n\}$, where $n \geq 3$. Players compete in a contest for increasing their shares of a given prize whose value is normalized to 1. We assume that a player $i \in N$ is either in a *friendly* relation or in an *antagonistic* relation with every other player in $N$. We write $F_i \subseteq N$ for the set of friends of $i$ including $i$ itself. As relations between friends (or enemies) are mutual, for any pair of players $i, j \in N$, we have $i \in F_j$ if and only if $j \in F_i$. We define a *network* as the profile of sets of friends $F := (F_1, ..., F_n)$ and we denote by $\mathcal{F}$ the set of all networks.[2]

Each player $i \in N$ is associated with an effort $x_i > 0$. We write $x = (x_1, ..., x_n) \in X \subseteq \mathbb{R}^n_{++}$ for the profile of efforts. For each player $i \in N$, we define a *success function* as a mapping $s_i : X \times \mathcal{F} \to (0, 1)$, which maps any effort profile and network pair $(x, F)$ into player $i$'s share $s_i(x, F)$. Note that we exclude the cases where efforts are zero or shares take value 0 or 1. These are marginal cases, but lead to complications in our characterization which can only be dealt with ad hoc axioms. In applications, whenever necessary, it seems natural to assume that $s_i(x, F) = 0$ if $x_i = 0$ and $x_j > 0$ for some $j \neq i$, so that resources are exclusively shared by players which actively participate.

We now define a particular class of success functions of interest. For each network $F \in \mathcal{F}$, effort profile $x \in X$ and player $i \in N$, this class is defined by the form

$$s_i^*(x, F) := \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)}, \tag{1}$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is any strictly increasing function. When we restrict our attention to all against all network where each player is friend with only itself, the class of success functions $s_i^*$ belongs to the one introduced by Skaperdas (1996). For contests between groups, $s_i^*$ does not immediately link to the class of success functions in Münster (2009) since it defines shares of individual players rather than groups of players. However we can easily relate two concepts by a simple transformation where the share of a group is defined by the sum of the shares of its members, so that $s_i^*$ defines a subclass of Münster (2009).[3]

We now show that the form (1) can be derived from basic assumptions via a standard probabilistic argument. For each $x \in X$, $F \in \mathcal{F}$ and $i \in N$, let the *effective strength* of player $i$ be

$$y_i(x, F_i) := \sum_{j \in F_i} g(x_j)$$

[2]In standard network theory, a network or a *graph* $\boldsymbol{g}$ is defined as a list of unordered pairs of players $\{i, j\}$ which are linked. Here instead, we define a network as a profile of sets of friends for convenience. We can easily link the two definitions in the following way: for each $F \in \mathcal{F}$ we define $\boldsymbol{g}$ such that $\{i, j\} \in \boldsymbol{g}$ if and only if $i \in F_j$. Note that this definition leads to non-directed graphs. We extend our model to weighted and directed graphs in Section 5.3.

[3]See Appendix for a formal discussion of how the two classes of functions relate.

where $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is some increasing function.[4] For each $x \in X$ and $F \in \mathcal{F}$, suppose that the share of player $i \in N$ satisfies

$$s_i(x, F) = \Pr\left(y_i(x, F_i) + \epsilon_i > y_j(x, F_j) + \epsilon_j \text{ for any } j \in N \backslash \{i\}\right), \quad (2)$$

where $(\epsilon_1, \ldots, \epsilon_n)$ are independent and identically distributed Gumbel random shocks. For simplicity let their variance be $\pi^2/6$, so that we get rid of a parameter in what follows. By a well-known result in McFadden (1973), condition (2) holds if and only if

$$s_i(x, F) = \frac{\exp\left(y_i(x, F_i)\right)}{\sum_{h \in N} \exp\left(y_i(x, F_h)\right)},$$

which can be rewritten as

$$s_i(x, F) = \frac{\prod_{j \in F_i} \exp\left(g(x_j)\right)}{\sum_{h \in N} \prod_{j \in F_h} \exp\left(g(x_j)\right)}. \quad (3)$$

One can easily see that (3) is equivalent to (1) if $f(x_i) = \exp\left(g(x_i)\right)$. Moreover, since $f > 1$, $\exp(0) = 1$ and the exponential function is monotonic, for any $g$ there is a unique $f$ such that $f(x_i) = \exp\left(g(x_i)\right)$ for all $x_i > 0$ (and for any $f$ there is a unique $g$ with $g(x_i) = \ln\left(f(x_i)\right)$ for all $x_i > 0$). Hence, (1) and (3) define the same class of functions. To summarize, we have shown that any success function from our class (1) is equivalent to the probability that the effective strength of a player is higher than anyone else's, in a context where effective strength takes an additive form and it is perturbed by random noise.[5]

Our class of success functions (1) proposes a simple environment to incorporate possible friendships in a conflict. Among the properties of $s_i^*$ we have that the sum of shares is always equal to 1, and the share of a player always increases in the effort of a friend and decreases in the effort of an enemy (as long as the player has at least one enemy).[6] It can be shown that, since $f > 1$, when a player becomes friend with a stronger player (a player with higher effort) its share always increases, while the share may decrease if the new friend is weaker. In the next section we characterize the class $s_i^*$ with a set of desirable properties of a model of conflict on networks.

[4]One can alternatively define the effective strength by subtracting from the above definition the sum of the efforts of enemies transformed by the function $g$. Since this would lead to the same class of functions (1), for simplicity we use the definition above.

[5]This probabilistic derivation is extended in Section 5.3 to the generalized model with weighted directed networks.

[6]To see this, it can be useful to rewrite (1) as

$$s_i^*(x, F) = \frac{\prod_{h \in F_i} \gamma(x_h) \prod_{k \notin F_i} \gamma(x_k)^{-1}}{\sum_{j \in N} \prod_{h \in F_j} \gamma(x_h) \prod_{k \notin F_j} \gamma(x_k)^{-1}}, \text{ where } \gamma(x_i) := \sqrt{f(x_i)}.$$

# 4 Characterization

In this section we give a characterization of our class of success functions (1) through six axioms. The first three are direct extensions of classical axioms in contest theory and they have similar justifications in our model. The latter three axioms incorporate the concept of friendships in conflict. The first condition, exhaustivity, requires that players always share the total of the prize.

*Exhaustivity*: For any $F \in \mathcal{F}$ and $x \in X$, $\sum_{i \in N} s_i(x, F) = 1$.

Anonymity states that shares are determined by efforts and networks, but not by players' identities. In short, it requires the contest to be a priori fair.

*Anonymity*: Let $\alpha$ be any permutation of $N$. For any $F \in \mathcal{F}$ and any $x \in X$, let $\alpha(F) = (F_{\alpha(1)}, ..., F_{\alpha(n)})$ and $\alpha(x) = (x_{\alpha(1)}, ..., x_{\alpha(n)})$. Then, $s_i(x, F) = s_{\alpha(i)}(\alpha(x), \alpha(F))$ for each $i \in N$.

We now impose two monotonicity axioms; namely, monotonicity of efforts and monotonicity of relations. Monotonicity of efforts imposes that the share of a player is strictly increasing in its effort. Monotonicity of relations implies that being friends with a stronger player increases the share.

*Monotonicity of efforts*: Let $F \in \mathcal{F}$ be any network with $F_i \neq N$ for some player $i \in N$ and $x \in X$ be any effort profile. Then, $s_i(x', F) > s_i(x, F)$ for any $x' \in X$ with $x'_i > x_i$ and $x'_k = x_k$ for all $k \neq i$.

*Monotonicity of relations*: Let $F \in \mathcal{F}$ be any network and $x \in X$ be any effort profile such that there is a pair $i, j \in N$ with $i \notin F_j$. Consider $F' \in \mathcal{F}$ such that $i \in F'_j$ and $F'_h = F_h$ for all $h \notin \{i, j\}$. Then, $s_i(x, F') > s_i(x, F)$ if $x_j > x_i$.

We finally introduce two axioms of independence. The first one, independence of efforts of commons, imposes that the ratio of the shares of two players (their *relative* share) is independent of the efforts of their common friends and common enemies.[7] The intuition is that the efforts of common friends and common enemies should similarly affect both players' shares; hence, they should not affect the relative share.

*Independence of efforts of commons (IEC)*: Take a network $F \in \mathcal{F}$ and two effort profiles $x, x' \in X$. For any pair of players $i, j \in N$, $\frac{s_i(x,F)}{s_j(x,F)} = \frac{s_i(x',F)}{s_j(x',F)}$ if $x_k = x'_k$ for all $k \in (F_i \cup F_j) \setminus (F_i \cap F_j)$.

All axioms above except monotonicity of relations impose conditions on a success function for a given network. How should we expect shares to change when net-

[7] We focus on the ratio of shares rather than, for instance, the difference of shares following a long tradition in probabilistic choice theory which dates back to the seminal work of Luce (1959) on independence of irrelevant alternatives. The axiomatic work in contest theory followed the same approach (see, e.g., Skaperdas, 1996). It is noteworthy that our axiom IEC can be alternatively formulated by imposing equivalence of any monotonic transformation of relative shares.

works change, i.e., when new friendships/enmities are made, besides an increase in the share upon making a stronger friend? The second independence axiom, independence of relations of others, focuses on the relative share of two players and identifies how the relative share should change when one of these players makes a new friend or enemy. More specifically, the axiom requires that the rate of change of this relative share remains the same across all pairs of networks which differ only by the new friendship. So, it can also be seen as a consistency requirement imposing the change that results from befriending a player to be consistent across networks.

*Independence of relations of others (IRO)*: Let $F \in \mathcal{F}$ be a network such that there are two players $i, j \in N$ with $i \in F_j$, and $x \in X$ be any effort profile. Let $F' \in \mathcal{F}$ be the network such that $i \notin F'_j$ and $F'_h = F_h$ for all $h \notin \{i, j\}$. Then, $\left(\frac{s_i(x,F')}{s_h(x,F')}\right) / \left(\frac{s_i(x,F)}{s_h(x,F)}\right) = \left(\frac{s_i(x,G')}{s_h(x,G')}\right) / \left(\frac{s_i(x,G)}{s_h(x,G)}\right)$ for all $h \in N \setminus \{i, j\}$ and $G, G' \in \mathcal{F}$ with $i \in G_j$, $i \notin G'_j$ and $G_k = G'_k$ for all $k \notin \{i, j\}$.

The following theorem states that our six axioms uniquely characterize the particular class of success functions $s_i^*$ defined by (1).

**Theorem 1** *A success function $s_i : X \times \mathcal{F} \to (0, 1)$ satisfies exhaustivity, anonymity, monotonicity of efforts, monotonicity of relations, IEC and IRO if and only if $s_i(x, F) = s_i^*(x, F)$ for any $i \in N$, $F \in \mathcal{F}$ and $x \in X$.*

Our characterization in Theorem 1 is tight. We demonstrate the tightness of the axioms by means of examples of success functions which satisfy all but one. Proofs are left to the reader.

1. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define $s_i^1(x, F)$ as

$$s_i^1(x, F) = \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h) + 1},$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is strictly increasing. This class of success functions satisfies all our axioms except exhaustivity.

2. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define $s_i^2(x, F)$ as

$$s_i^2(x, F) = \frac{\prod_{h \in F_i} f_h(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f_h(x_h)},$$

where $f_i : \mathbb{R}_{++} \to (1, +\infty)$ is strictly increasing for each $i \in N$ and $f_h \neq f_j$ for some $h, j \in N$. This class of success functions fulfills all our axioms except anonymity.

3. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define $s_i^3(x, F)$ as

$$s_i^3(x, F) = \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)},$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is strictly decreasing. Then, $s_i^3$ satisfies all axioms except monotonicity of efforts.

4. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define $s_i^4(x, F)$ as

$$s_i^4(x, F) = \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)},$$

where $f : \mathbb{R}_{++} \to (0, 1)$ is strictly increasing. Then, $s_i^4$ fulfills all axioms but monotonicity of relations.

5. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, let us define $s_i^5(x, F)$ as

$$s_i^5(x, F) = \frac{\exp\left(x_i / \sum_{k \in N} x_k\right) \prod_{h \in F_i} \exp(x_h)}{\sum_{j \in N} \exp\left(x_j / \sum_{k \in N} x_k\right) \prod_{h \in F_j} \exp(x_h)}.$$

This success function violates IEC, while it satisfies all other axioms.

6. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, let us define $s_i^6(x, F)$ as

$$s_i^6(x, F) = \begin{cases} \dfrac{f(x_i)^2}{\sum_{j \in N} f(x_j)^2} & \text{if } F_k = \{k\} \text{ for all } k \in N, \\ \dfrac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)} & \text{otherwise,} \end{cases}$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is a strictly increasing function. Then, $s_i^6$ fulfills all axioms but IRO.

Among these six examples, it is easy to see that $s_i^1$ and $s_i^2$ are characterized by the five axioms they satisfy. On the other hand, $s_i^3$ and $s_i^4$ can easily be characterized by further imposing the 'opposite' of the monotonicity axiom that each fails to satisfy. Characterization of $s_i^5$ and $s_i^6$ requires introducing other axioms and is not necessarily desirable.

# 5 Extensions and applications

Due to its novelty and generality, our framework has potential for many applications. In Section 5.1 and Section 5.2, we illustrate two examples. First, we consider games where players choose relations taking efforts as given. Then, we study games where players choose efforts taking relations as given.

We have so far assumed that the friendship between two players is mutual and that the degree of friendliness is the same across friends. Our setting belongs to a broader class of models of conflict on networks where each friendship might have a different weight and is not necessary mutual. In Section 5.3, we extend our model by introducing weighted directed friendships and derive a class of success functions via a probabilistic argument. It turns out that the class defined by $s_i^*$ corresponds to this class when we limit our attention to unweighted undirected friendships.

## 5.1 Application: network formation problems

In these games, players take their efforts as given and simultaneously choose whether to become friend with each other player or not. This setting belongs to the literature on network formation. See Jackson (2005) for a review on network formation games. These games are suitable for applications in international relations. Each player can represent a country whose effort is a measure of its military capability. While military capability is a stock, relations can change at convenience due to their ephemeral nature.

Our framework can be seen as a specific network formation problem where all networks generate the same value which is 1; hence the value function is constant. The success function works as the allocation rule. Then, for this subsection we define the payoff of a player $i \in N$ simply by its share of the prize $s_i$, hence $\pi_i(x, F) = s_i(x, F)$ for any $x \in X$ and $F \in \mathcal{F}$. A common solution concept in this literature is *pairwise stability.* To formally define pairwise stability, we introduce some notation. Given a network $F \in \mathcal{F}$ and a pair of players $i, j \in N$ with $i \in F_j$, let $F - ij$ be the network where players $i$ and $j$ become enemies while all other relations remain as in $F$. Similarly, given a network $F \in \mathcal{F}$ and a pair of players $i, j \in N$ with $i \notin F_j$, let $F + ij$ be the network where players $i$ and $j$ become friends while all other relations remain as in $F$.

**Definition 1** *For a given effort profile $x \in X$, a network $F \in \mathcal{F}$ is pairwise stable if*

**(i)** *for all $i, j \in N$ with $i \in F_j$, $\pi_i(x, F) > \pi_i(x, F - ij)$ and $\pi_j(x, F) > \pi_j(x, F - ij)$ and*

**(ii)** *for all $i, j \in N$ with $i \notin F_j$, if $\pi_i(x, F) \leq \pi_i(x, F + ij)$ then $\pi_j(x, F) \geq \pi_j(x, F + ij)$.*

In a pairwise stable network, no pair of friends can both be better off by breaking their friendship and no pair of enemies can both be better off by becoming friends. We now provide a result which characterizes the unique pairwise stable network when efforts are symmetric.

**Proposition 1** *Let the success function take the form $s_i^*$. For any symmetric effort profile $x \in X$, the unique pairwise stable network is the peace network where all players are friends.*

In the context of international relations, our result simply means that if all countries had equal military capabilities, peace for all would be the only pairwise stable network.

## 5.2 Application: contest games

In contest games, players take the network of relations as given and simultaneously choose their efforts. This is the standard setting in contest theory if we restrict our attention to the all against all network. Let us describe the game in more detail. Following common assumptions in the literature, for any network $F \in \mathcal{F}$ and effort profile $x \in X$, in this subsection we define the payoff of player $i \in N$ as $\pi_i(x, F) = s_i(x, F) - cx_i$, where $c > 0$ is the marginal cost of effort. Each player $i \in N$ chooses $x_i$ to maximize $\pi_i(x, F)$ and we solve for Nash equilibrium of the corresponding game. We state a simple result for a network where all players have equal number of friends.

**Proposition 2** *Let the success function take the form $s_i^*$ with $f(x_i) = 1 + x_i$. For any network $F \in \mathcal{F}$ where all players have the same number of friends $k < n$, there exists an equilibrium $x^* \in X$ such that for all $i \in N$ $x_i^* = (n-k)/(n^2c) - 1$ if $c < 1/n^2$.*

This simple result shows that equilibrium efforts decrease in the number of friends $k$. This setting is suitable for applications to political lobbying among others. Let us describe a simple hypothetical example of political lobbying. Suppose interests groups are defined according to the attributes their members possess. As an example, consider four interest groups defined by all combinations of native/immigrant and woman/man. There is a fixed public budget and each group lobbies for a larger share. The group of immigrant women lobbies for transfers to all immigrants and all women, while indirectly helping immigrant men and native women in their lobbying activities. This implies an informal alliance between groups that share one attribute. On the other hand, native men always lobby in the opposite direction to immigrant women. See Figure 2 for the network that represents these relations. Since relations between interest groups are based on origin and gender, they are unlikely or impossible to change and the setting we describe above applies.

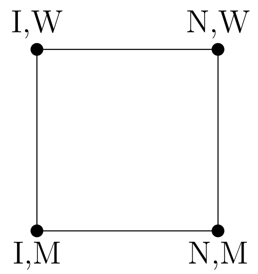


Figure 2: Political lobbying: N,W denotes the group "native woman" and other groups are analogously labeled.

Let us take each player as an individual rather than a group for simplicity. Every player has the same number of friends which is 3 including itself. If $f(x_i) = 1 + x_i$ and $c < 1/n^2$, equilibrium efforts are $1/(16c) - 1$ as $n = 4$ and $k = 3$. We now compare this result with another one for the alternative setting where both women are native and both men are immigrants. This gives a different network, where each player has 2 friends including itself (native women are friends with each other and immigrant men are friends with each other). The corresponding equilibrium efforts are $1/(8c) - 1$ by Proposition 2. Since $1/(16c) - 1 < 1/(8c) - 1$ for any $c < 1/n^2$, our success function predicts more effort – more lobbying – in the second case, when ethnic origin and gender are perfectly correlated leading to a more polarized set of players compared to the first setting.

This is of course a very simple example but this analysis can be extended to more complex networks to study, for instance, how the intensity of political lobbying in a country is affected by the structure of the identities of citizens. This approach would be in line with Esteban and Ray (1999) and Esteban et al. (2012), which show the positive link between ethnic polarization and conflict from a theoretical and empirical view point respectively.

## 5.3 Extension: weighted directed networks

For this subsection, we allow for varying degrees of friendship across players and we extend the probabilistic derivation in Section 3 to weighted directed networks. The intensity of friendship that player $i$ receives from player $j$ is given by $\phi_{i,j} \in [0,1]$. We denote by $\phi_i := (\phi_{i,1}, \ldots, \phi_{i,n})$ the vector of intensities of all friendships from all players to player $i$ (including $i$ itself). The profile of friendships $\phi = (\phi_1, \ldots, \phi_n)$ is a matrix defining a weighted directed network of relations. We define the *generalized success function* as the mapping $p_i : X \times \Phi \to (0,1)$ which determines the share of player $i$ given the effort profile $x$ and the friendship profile $\phi \in \Phi := [0,1]^{n\times n}$. For any $x \in X$ and $\phi \in \Phi$, we impose the share of player $i \in N$ to satisfy

$$p_i(x,\phi) = \Pr\left(w_i(x,\phi_i) + \epsilon_i > w_j(x,\phi_j) + \epsilon_j \text{ for any } j \in N\setminus\{i\}\right), \tag{4}$$

where $w_i(x,\phi_i) := \sum_{j\in N} \phi_{i,j} g(x_j)$ is the effective strength of each player $i \in N$ given any increasing function $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$, and $(\epsilon_1, \ldots, \epsilon_n)$ are independent and identically distributed Gumbel random shocks with variance $\pi^2/6$. Following the same arguments in Section 3, (4) holds if and only if

$$p_i(x,\phi) = \frac{\prod_{j\in N} \exp\left(\phi_{i,j} g(x_j)\right)}{\sum_{h\in N}\prod_{j\in N} \exp\left(\phi_{h,j} g(x_j)\right)}. \tag{5}$$

Then, (5) defines our class of generalized success functions, where the only restriction on $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is that it is increasing. Moreover, one can easily see the relationship between (5) and (3): if we restrict $\phi_{i,j}$ to take value only in $\{0,1\}$ for every $i,j \in N$ (where 0 means enmity and 1 means friendship) and $\phi_{i,j} = \phi_{j,i}$, the model reduces to the one in Section 3, and (5) reduces to (3). Our axiomatization of the benchmark case (3) also provides a deeper understanding of (5). Immediate extensions of exhaustivity, anonymity and the two monotonicity axioms apply in the setting of weighted directed friendships; though, it is not straightforward to see how the two independence axioms can be generalized. Since the network variable is new in the axiomatic work on contest success functions, it is worth starting simple and we leave the axiomatization of the generalized class to future research.

# 6 Conclusion

We define and axiomatically characterize a class of success functions for contests where each pair of players can be in a friendly or in an antagonistic relation, and these pairwise relations form a network. For all against all contests, our class of success functions belongs to the one characterized in Skaperdas (1996). For contests between groups, the aggregate share of group members calculated by our functional form always belongs to the class axiomatized in Münster (2009). In our basic setup a success function treats every friend equally. However, it can easily be extended to the setting where each friendship has a weight and relations between players are not necessarily mutual. We define this more general class of models in Section 5.3 following a probabilistic argument in McFadden (1973). We leave axiomatic characterization of this generalized form for future research.

Our framework allows to study strategic interaction between parties in conflict who are connected by a complex network of relations. Among many other environments, we commonly see such complex networks in international relations between countries, in industrial organization between competing firms and in political lobbying between interest groups. We illustrate two possible applications of our model. In Section 5.1, we show that under symmetric efforts the unique pairwise stable network is the peace network when our success function works as the allocation rule in the corresponding network formation problem. A very natural question that arises is whether this result persists under asymmetric effort profiles. We know that it does not persist generally and we leave the characterization of such profiles for future research. Similarly, it is interesting to extend the analysis in Section 5.2 to any type of networks, or to study an environment where players choose their efforts and friendships simultaneously.

Our success function has potential for empirical applications as well. For instance, our model can be used to test for network effects by maximum likelihood or related methods. We refer to Jia et al. (2013) for a review of empirical issues in the estimation of success functions. The success function can also be used as an index of power adjusted for the network of relations. In the context of international relations, a particularly suited collection of datasets is presented by The Correlates of War Project, which spans for about two centuries and provides material for case studies as well as econometric analysis. In this context, the effort of a country can be estimated by its National Material Capabilities (see Singer et al., 1972) while its set of friends can be estimated by its Formal Alliances (see Gibler, 2009).

# Appendix

### Relation to Münster (2009)'s class of success functions

We now show how our class of success functions (1) relates to the one in Münster (2009) by applying a simple transformation. Let $\tilde{\mathcal{F}} \subset \mathcal{F}$ be the set of all networks which organize players into mutually exclusive groups, i.e., coalitions. Denote by $\mathcal{C}(F)$ the partition of $N$ defined by network $F \in \tilde{\mathcal{F}}$. Münster (2009) shows that under some conditions the share of group $C \in \mathcal{C}(F)$ must take the form $\sigma_C(x, F) := \varphi_C(x_C)/\left[\sum_{C' \in \mathcal{C}(F)} \varphi_{C'}(x_{C'})\right]$ for any $x \in X$ and $F \in \tilde{\mathcal{F}}$, where for any group $C \in \mathcal{C}(F)$ the vector $x_C$ defines the efforts of all members of $C$ and the function $\varphi_C : \mathbb{R}^{|C|}_{++} \to \mathbb{R}_{++}$ increases in all its arguments. Let $\sigma^*_C(x, F) := \sum_{i \in C} s^*_i(x, F)$. By (1), it is straightforward that $\sigma^*_C(x, F)$ belongs to the class of Münster (2009) for $\varphi_C(x_C) = |C| \prod_{i \in C} f(x_i)$.

□

### Proof of Theorem 1

We leave to the reader to verify that (1) satisfies the axioms given in the theorem. To show that the converse holds, for each player $i \in N$, we take a success function $s_i : X \times \mathcal{F} \to (0, 1)$ satisfying the axioms. We want to show that for any $i \in N$, $x \in X$ and $F \in \mathcal{F}$, $(*)$ $s_i(x, F) = s^*_i(x, F)$.

Let $F \in \mathcal{F}$ be the network for which $F_i = N$ for each player $i \in N$. Take any effort profile $x \in X$. To show that $(*)$ is true, it suffices to show that $(**)$ $s_i(x, F) = 1/n$. Take any pair $i, j \in N$. Take a permutation $\alpha$ such that $\alpha(i) = j$, $\alpha(j) = i$, $\alpha(k) = k$ for all $k \notin \{i, j\}$. By anonymity, $s_i(x, F) = s_j(\alpha(x), F)$ and $s_j(x, F) = s_i(\alpha(x), F)$. Note that $\alpha(F) = F$. Moreover, by IEC $\frac{s_i(x,F)}{s_j(x,F)} = \frac{s_i(\alpha(x),F)}{s_j(\alpha(x),F)}$ which then is equal to $\frac{s_j(x,F)}{s_i(x,F)}$. This implies that $s_i(x, F) = s_j(x, F)$. Together with exhaustivity this implies $(**)$.

To proceed in the proof, we now define a class of success functions slightly broader than (1). As an intermediate result we will argue that the axioms require the success function to belong to this class. For any $F \in \mathcal{F}$, $x \in X$ and $i \in N$, we define

$$\hat{s}_i(x, F) = \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)} \tag{6}$$

where $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$. Note that (6) differs from (1) only by $f$ not necessarily being strictly increasing and strictly greater than 1. Moreover it is easy to verify that (6) satisfies all our axioms except monotonicity of efforts and monotonicity of relations. Hence, (6) defines a class of success functions which includes (1).

To prove our intermediate result, we proceed by induction. First, let $F \in \mathcal{F}$ be a network with at least one pair of players which are enemies, i.e., $F_i \neq N$ for at least one player $i$. Let $x \in X$ be an effort profile. It is easy to show that there exists a sequence of networks $F^0, \ldots, F^m$ with $m \geq 1$ such that for $t \in \{0, \ldots, m-1\}$ (i) there is a pair of players $i, j \in N$ such that $i \in F_j^t$ and $i \notin F_j^{t+1}$, (ii) for all $k \notin \{i, j\}$, $F_k^t = F_k^{t+1}$ and (iii) $F_i^0 = N$ for all $i \in N$ and $F^m = F$. Note that there are $(m-1)!$ such sequences. We take any such sequence and we want to prove the following claim:

*Claim*: For each $i \in N$, $s_i(x, F^t) = \hat{s}_i(x, F^t) \implies s_i(x, F^{t+1}) = \hat{s}_i(x, F^{t+1})$.

Let $s_k(x, F^t) = \hat{s}_k(x, F^t)$ for all $k \in N$. Without loss of generality, we take two players $i, j \in N$ with $i \in F_j^t$ and we define the network $F^{t+1}$ as the network where $i \notin F_j^{t+1}$ and $F_h^{t+1} = F_h^t$ for all $h \notin \{i, j\}$. Let $h$ be any player in $N \backslash \{i, j\}$. By IEC, $\frac{s_i(x,F^t)}{s_h(x,F^t)}$ depends only on the efforts of the uncommon friends of $i$ and $h$ in $F^t$. Similarly, $\frac{s_i(x,F^{t+1})}{s_h(x,F^{t+1})}$ depends only on the efforts of their uncommon friends in $F^{t+1}$. Hence, there exists a real valued function $\gamma_{i,j,h}^{F^t}$ such that

$$\gamma_{i,j,h}^{F^t}(x_U) = \left( \frac{s_i(x, F^{t+1})}{s_h(x, F^{t+1})} \right) / \left( \frac{s_i(x, F^t)}{s_h(x, F^t)} \right) \tag{7}$$

where $x_U$ is the profile of efforts of players $l \in U := [(F_i^t \cup F_h^t) \setminus (F_i^t \cap F_h^t)] \cup [(F_i^{t+1} \cup F_h^{t+1}) \setminus (F_i^{t+1} \cap F_h^{t+1})]$. By IRO, for any pair of networks $G, G' \in \mathcal{F}$ such that $i \in G_j$ and $i \notin G'_j$ and $G'_k = G_k$ for all $k \notin \{i, j\}$,

$$\gamma_{i,j,h}^{F^t}(x_U) = \left( \frac{s_i(x, G')}{s_h(x, G')} \right) / \left( \frac{s_i(x, G)}{s_h(x, G)} \right). \tag{8}$$

By IEC the right hand side of (8) is exclusively a function of $x_{U'}$, where $U' := [(G_i \cup G_h) \setminus (G_i \cap G_h)] \cup [(G'_i \cup G'_h) \setminus (G'_i \cap G'_h)]$. Note that $j \in U$ and $j \in U'$ by construction. As there is no restriction on $G$ except that $i \in G_j$, the function $\gamma_{i,j,h}^{F^t}$ does not depend on the whole network $F^t$ but only on the relation between $i, j$. Moreover, as (8) must hold for $G$ such that $U' = \{j\}$, the function $\gamma_{i,j,h}^{F^t}$ is constant in all efforts except $x_j$. A similar set of expressions can be written for the relative share of players $j$ and $h$ in networks $F^t$ and $F^{t+1}$. Then, we can define the functions $g_{i,j,h} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $g_{j,i,h} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ so that

$$g_{i,j,h}(x_j) = \left( \frac{s_i(x, F^{t+1})}{s_h(x, F^{t+1})} \right) / \left( \frac{s_i(x, F^t)}{s_h(x, F^t)} \right), \tag{9}$$

$$g_{j,i,h}(x_i) = \left( \frac{s_j(x, F^{t+1})}{s_h(x, F^{t+1})} \right) / \left( \frac{s_j(x, F^t)}{s_h(x, F^t)} \right). \tag{10}$$

For $n = 3$, we can immediately write $g_{i,j,h}(x_j) = g_{i,j}(x_j)$ and $g_{j,i,h}(x_i) = g_{j,i}(x_i)$ for the unique player $h \notin \{i, j\}$. Let $n \geq 4$, so that there are at least two players $h, k \in N \setminus \{i, j\}$. If we write (9) for $h, k \in N \setminus \{i, j\}$ and we take the ratio of the two expressions we obtain

$$\frac{g_{i,j,k}(x_j)}{g_{i,j,h}(x_j)} = \left( \frac{s_h(x, F^{t+1})}{s_k(x, F^{t+1})} \right) / \left( \frac{s_h(x, F^t)}{s_k(x, F^t)} \right).$$

Similarly if we write (10) for $h, k \in N \setminus \{i, j\}$ we obtain

$$\frac{g_{j,i,k}(x_i)}{g_{j,i,h}(x_i)} = \left(\frac{s_h(x, F^{t+1})}{s_k(x, F^{t+1})}\right) / \left(\frac{s_h(x, F^t)}{s_k(x, F^t)}\right).$$

Let $G, G' \in \mathcal{F}$ be such that $i \in G_j$ and $i \notin G'_j$ and $G'_l = G_l$ for all $l \notin \{i, j\}$. Moreover let $G_h = G_k$. Consider the permutation $\beta$ such that $\beta(k) = h$, $\beta(h) = k$, $\beta(l) = l$ for all $l \notin \{h, k\}$. By anonymity $s_h(x, G) = s_k(\beta(x), \beta(G))$. Note that $\beta(G) = G$, therefore $s_h(x, G) = s_k(\beta(x), G)$. As $G_h = G_k$ implies $G'_h = G'_k$, by anonymity we must also have $s_h(x, G') = s_k(\beta(x), G')$. Moreover, by IEC $\frac{s_h(x,G)}{s_k(x,G)} = \frac{s_h(\beta(x),G)}{s_k(\beta(x),G)}$ which then is equal to $\frac{s_k(x,G)}{s_h(x,G)}$. This implies that $s_h(x, G) = s_k(x, G)$. Similarly, by IEC we also have $s_h(x, G') = s_k(x, G')$. It follows by IRO that

$$1 = \left(\frac{s_h(x, F^{t+1})}{s_k(x, F^{t+1})}\right) / \left(\frac{s_h(x, F^t)}{s_k(x, F^t)}\right), \tag{11}$$

hence $g_{i,j,h}(x_j)$ does not depend on the identity of $h$ as long as $h \notin \{i, j\}$. Then, we can write $g_{i,j,h}(x_j) = g_{i,j}(x_j)$ and $g_{j,i,h}(x_i) = g_{j,i}(x_i)$ for each $h \in N \setminus \{i, j\}$ also when $n \geq 4$.

Now, let $G' \in \mathcal{F}$ be the network such that $G'_k = \{k\}$ for all $k \in N$. Consider the network $G \in \mathcal{F}$ which differs from $G'$ only by $i, j$ being friends, so $i \in G_j$ and $G_k = G'_k$ for all $k \notin \{i, j\}$. Consider the permutation $\alpha$ defined above. By anonymity $s_i(x, G) = s_j(\alpha(x), \alpha(G))$. Note that $\alpha(G) = G$, therefore $s_i(x, G) = s_j(\alpha(x), G)$. Then, as by IEC $\frac{s_i(x,G)}{s_j(x,G)}$ is constant in $x$, we must have $s_i(x, G) = s_j(x, G)$. Using IRO, we can write

$$\left(\frac{s_i(x, G')}{s_j(x, G')}\right) / \left(\frac{s_i(x, G)}{s_j(x, G)}\right) = \frac{s_i(x, G')}{s_j(x, G')} = \frac{g_{i,j}(x_j)}{g_{j,i}(x_i)}. \tag{12}$$

Since $\alpha(G') = G'$, by anonymity $g_{i,j} = g_{j,i}$. Note that any permutation of players besides $\alpha$ also leads to $G'$. Then, anonymity implies that $g_{i,j}$ and $g_{j,i}$ do not depend on the identities of $i$ and $j$, hence we can write $g_{i,j} = g_{j,i} = g$ for all $i, j \in N$.

Let $s_i(x, F^t) = \hat{s}_i(x, F^t)$ for all $i \in N$, $x \in X$ and some $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$. By (9) we can write

$$g(x_j) = \left(\frac{s_i(x, F^{t+1})}{s_h(x, F^{t+1})}\right) / \left(\frac{\hat{s}_i(x, F^t)}{\hat{s}_h(x, F^t)}\right),$$

therefore

$$\frac{s_i(x, F^{t+1})}{s_h(x, F^{t+1})} = \frac{\prod_{l \in F_i^t} f(x_l) g(x_j)}{\prod_{l \in F_h^t} f(x_l)}. \tag{13}$$

To show that the Claim holds, we have to show that $g = 1/f$. Suppose for a contradiction, this is not the case. Consider the specific case where $\hat{F}^t \in \mathcal{F}$ be the network where $i \in \hat{F}_j^t$ and $\hat{F}_k^t = \{k\}$ for all $k \in N \setminus \{i, j\}$. Let $\hat{F}^{t+1}$ be defined accordingly. If we rewrite (13) for this pair of networks, we obtain

$$\frac{s_i(x, \hat{F}^{t+1})}{s_h(x, \hat{F}^{t+1})} = \frac{f(x_i) f(x_j) g(x_j)}{f(x_h)}, \tag{14}$$

which equals $\frac{g(x_h)}{g(x_i)}$ by (12). Then, we can write

$$\frac{f(x_i)f(x_j)g(x_j)}{f(x_h)} = \frac{g(x_h)}{g(x_i)}. \tag{15}$$

As the right hand side is constant in $x_j$, we must have $f(x_j)g(x_j) = c$ where $c$ is a strictly positive constant. Plugging $c/f$ for $g$ in (15) and rearranging, we have

$$\frac{f(x_i)f(x_j)}{f(x_h)} = \frac{c/f(x_h)}{(c/f(x_i))(c/f(x_j))}, \tag{16}$$

which implies that $c = 1$, hence a contradiction. Therefore $g = 1/f$ and we can rewrite (13) as

$$\frac{s_i(x, F^{t+1})}{s_h(x, F^{t+1})} = \frac{\prod_{l \in F_i^{t+1}} f(x_l)}{\prod_{l \in F_h^{t+1}} f(x_l)}. \tag{17}$$

By (12) and (17), for any $i, j \in N$ we have

$$\frac{s_i(x, F^{t+1})}{s_j(x, F^{t+1})} = \frac{\prod_{l \in F_i^{t+1}} f(x_l)}{\prod_{l \in F_j^{t+1}} f(x_l)},$$

hence all relative shares are determined. For any $j \in N$, by exhaustivity,

$$\frac{1}{s_j(x, F^{t+1})} = \sum_{i \in N} \frac{s_i(x, F^{t+1})}{s_j(x, F^{t+1})} = \sum_{i \in N} \left( \frac{\prod_{l \in F_i^{t+1}} f(x_l)}{\prod_{l \in F_j^{t+1}} f(x_l)} \right) = \frac{\sum_{i \in N} \prod_{l \in F_i^{t+1}} f(x_l)}{\prod_{l \in F_j^{t+1}} f(x_l)}$$

therefore $s_j(x, F^{t+1}) = \hat{s}_j(x, F^{t+1})$. Then, we have shown that the Claim holds.

As (6) satisfies all axioms except monotonicity of efforts and monotonicity of relations, it can be proven that $s_i(x, F^0) = \hat{s}_i(x, F^0)$ for any positive function $f$. Then, for any network $F \in \mathcal{F}$ with at least one pair of players which are enemies, any effort profile $x \in X$, $s_i(x, F) = \hat{s}_i(x, F)$ for any $i \in N$ by induction.

Given this, to prove (∗) for such networks and effort profiles it is sufficient to show that the function $f$ must always be strictly increasing and strictly greater than 1. Consider a network $F \in \mathcal{F}$ such that there is a pair $i, j \in N$ with $i \notin F_j$. As $i \notin F_j$, we have $s_i(x, F) = \hat{s}_i(x, F)$ and $s_j(x, F) = \hat{s}_j(x, F)$ for any $x \in X$. Take any $x, x' \in X$ such that $x'_i > x_i$ and $x'_h = x_h$ for $h \neq i$. By monotonicity of efforts $s_i(x', F) > s_i(x, F)$. Then, it is easy to verify that $f$ must be strictly increasing. To show that $f > 1$, consider the network $F'$ with $i \notin F'_j$ and $F'_k = F_k$ for all $k \notin \{i, j\}$. Without loss of generality, let $x_j > x_i$. Then, $s_i(x, F) > s_i(x, F')$, which implies that $f > 1$. So, we achieve the desired result (∗).

□

**Proof of Proposition 1**

Take any pair of players $i, j \in N$ and a network $F \in \mathcal{F}$ where $i, j$ are friends. Let $F'$ be the network in $\mathcal{F}$ where $i, j$ are enemies and all other relations are the same as in $F$. For any effort profile $x \in X$, one can show that we have $s_i(x,F) > s_i(x,F')$ if and only if

$$f(x_i) - f(x_j) < (f(x_j) - 1) \left( \sum_{h \in N \setminus \{i,j\}} \prod_{k \in F_h} f(x_k) \right) / \left( \prod_{h \in F_j} f(x_h) \right). \quad (18)$$

Suppose the effort profile $x$ is symmetric, i.e., $x_1 = \ldots = x_n > 0$. The LHS of (18) is always 0. Conversely the RHS is always positive. Then, by (18), any network $F'$ where some players $i, j \in N$ are enemies is not pairwise stable, as they prefer to be friends. It follows that the only pairwise stable network is the one where all players are friends.

□

**Proof of Proposition 2**

Take any $F \in \mathcal{F}$ such that $k < n$. It is easy to verify that, if $f(x_i) = 1 + x_i$, the second derivative of $\pi_i(x,F)$ with respect to $x_i$ is negative for any $x \in X$ and $i \in N$. Then, a solution $x^*$ of the system of equations given by the $n$ first order conditions

$$s_i(x,F) \sum_{j \notin F_i} s_j(x,F) = cf(x_i)/f'(x_i)$$

of all players $i \in N$ defines a Nash equilibrium. Suppose that a solution satisfies $x_i^* = x_j^*$ for any $i, j \in N$. As $|F_i| = k$ for any $i \in N$, we have $s_i(x^*,F) = 1/n$ for all $i \in N$ and each first order condition becomes $(n-k)/n^2 = cf(x_i^*)/f'(x_i^*)$. As $f(x_i) = 1 + x_i$, we have $x_i^* = (n-k)/(n^2 c) - 1$. Note that $x_i^* > 0$ for all $k < n$ if and only if $c \in (0, 1/n^2)$. Then, $x^*$ is an equilibrium if and only if $c \in (0, 1/n^2)$.

□

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