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**JUAN PABLO RINCÓN ZAPATERO**  
*(Universidad Carlos III de Madrid)*

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# Recursive Utility and Thompson Aggregators\*

Robert A. Becker  
Department of Economics  
Indiana University  
Bloomington, IN 47405  
USA

Juan Pablo Rincón-Zapatero  
Departamento de Economía  
Universidad Carlos III de Madrid  
28903 Getafe (Madrid)  
Spain

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## Abstract

We reconsider the theory of Thompson aggregators proposed by Marinacci and Montrucchio. First, we prove a variant of their Recovery Theorem establishing the existence of extremal solutions to the Koopmans equation. Our approach applies the constructive Tarski-Kantorovich Fixed Point Theorem rather than the nonconstructive Tarski Theorem employed in their paper. We verify the Koopmans operator has the order continuity property that underlies invoking Tarski-Kantorovich. Then, under more restrictive conditions, we demonstrate there is a unique solution to the Koopmans equation. Our proof is based on  $u_0$ -concave operator techniques as first developed by Kransosels'kii. This differs from Marinacci and Montrucchio's proof as well as proofs given by Martins-da-Rocha and Vailakis.

JEL Codes: D10, D15, D50,E21

Keywords: Recursive Utility, Thompson Aggregators, Koopmans Equation, Extremal Solutions, Concave Operator Theory

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# 1 Introduction

Recursive utility functions defined for discrete time, deterministic, and infinite horizon intertemporal choice problems have been studied intensively since their introduction by Koopmans ([23], [24], and [25]). Koopmans, Diamond and Williamson [26] extended that work. Koopmans showed a recursive utility function satisfied a particular functional equation, known today as the *Koopmans equation*. This equation relates the utility function to an *aggregator function* in two real variables, current consumption and future utility.

Lucas and Stokey [32] proposed taking the aggregator as the primitive concept. Using that function, the *Koopmans equation* in the unknown utility function is defined and a unique solution (in an appropriate function space) is sought. This solution *recovers* a unique recursive utility function representation of the underlying preference relation defined on the commodity space. This existence and uniqueness problem is solved by setting up a fixed point problem for the *Koopmans operator*. It is a selfmap defined on the given space of functions considered as potential utility functions representing the underlying preference relation. They appeal to Banach's Contraction Mapping Principle. In order to do so, the aggregator function must be carefully restricted in order to prove the Koopmans operator satisfies Blackwell's sufficient condition for a contraction mapping. Generalizations of the Lucas and Stokey approach are the subject of Boyd [12] and the monograph by Becker and Boyd [9]. Their work lays out the *recovery* theory for the class of *Blackwell aggregators*.

Subsequent research initiated by Marinacci and Montrucchio [37] introduced the new line of *Thompson aggregators* to distinguish them from the Blackwell aggregators. The motivation for this new class is that economically reasonable aggregators can fail to satisfy the key contraction property required by various forms of contraction theorems in proving the existence and uniqueness of a solution to the Koopmans equation. There are two sources for aggregators that fail to satisfy the conditions for Blackwell aggregators. The main issue is the Lipschitz condition. In the first class of aggregators the Lipschitz condition on future utility fails altogether.<sup>1</sup> In the second class, a Lipschitz condition holds, but corresponds to upcounting, or possibly no discounting of future utility.<sup>2</sup> In both situations the contraction property breaks down. New techniques must be introduced in order to associate utility functions with these so-called Thompson aggregators.

Blackwell's sufficient condition for a contraction mapping assumes for recursive utility applications the Koopmans operator is a monotone self map. This property alone is sufficient, in many examples, to prove the existence of extremal solutions in the stated function space. This approach was taken by Marinacci and Montrucchio [37] since it is easy to verify monotonicity of the Koopmans operator for Thompson aggregators. They go on to separate the question of

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<sup>1</sup>In this case the interesting contribution by Le Van and Vailakis [30] does not apply.

<sup>2</sup>This case may overlap with the aggregator conditions in Le Van and Vailakis [30]. However, they also impose a strong condition, their assumption (W5), that might fail for a Thompson aggregator.

existence of a solution to the Koopmans equation from the determination of whether or not that solution is unique in the given space of possible utility representations. Their existence proof turns on an application of the well-known Tarski Fixed Point Theorem that obtains a smallest and largest fixed point of the Koopmans operator. These fixed points are the *extremal fixed points*. Marinacci and Montrucchio [37] define an underlying space of possible utility functions that is an order interval in a space of bounded functions forming a Banach lattice. Their order interval is a complete lattice. However, it is well known that Tarski's Theorem is nonconstructive. The iterative scheme they provide for "computing" the extremal fixed points by successive approximation may fail to yield those extremal solutions. The missing ingredient is the requirement that the Koopmans operator enjoy an order continuity property. Our paper verifies this property holds in their setting. Absent such a proof, the extremal fixed points may only be found through transfinite induction. Although this is an iterative procedure, it is hardly a constructive one. Hence, it is desirable to prove a constructive version of their result in order to provide a foundation for computing approximate solutions to the Koopmans equation derived from Thompson aggregators. The notion of a constructive procedure as used here means use of successive approximations indexed on the natural numbers.

There is another advantage to our constructive approach that is noted by Marinacci and Montrucchio [37], if not clearly justified in their paper — the smallest fixed point turns out to be sup norm lower semicontinuous and the largest one is sup norm upper semicontinuous as real-valued functions defined on the underlying commodity space. The order continuity property of the Koopmans operator also implies its set of fixed points is a countably chain complete partially ordered set. This property, first demonstrated in Balbus, Reffett, and Woźny [6], is the constructive analog of Tarski's proof that the set of fixed points is a complete lattice in its own right.

We begin with a brief review of concepts on partially ordered sets, lattices, and positive cones in real Banach spaces. Next we recall the Tarski-Kantorovich Theorem and related concepts. The aggregator axioms and basic theory derived from Marinacci and Montrucchio [37] follow next. The fourth section includes our version of the Marinacci and Montrucchio existence theorem, which we term a *recovery theorem*. Next we turn to the uniqueness problem. The issue is to show that the two extremal fixed points found in the Tarski-Kantorovich approach are equal, implying there is exactly one solution to the Koopmans equation on the specified order interval and it is sup norm continuous. We prove our result as an application of  $u_0$ -concave operator theory, a stronger form of a concave operator. This argument may have independent interest as we draw on new results obtained by Liang et al [31] that provide sufficient conditions for a  $u_0$ -concave operator acting on the positive cone of a Banach space to admit at most one fixed point. This part of our paper restricts the commodity space to the positive cone of the space of all bounded real-valued sequences. The existence theory, by contrast, admits commodity spaces offering sustained growth (modeled as principal ideals in the space of all real-valued sequences). We conclude with some thoughts for future research and comments

on the necessity to extend our results to searching for continuous solutions to the Koopmans equation with weaker topologies than the norm topologies featured here.<sup>3</sup>

## 2 Mathematical Preliminaries

### 2.1 Posets, Lattices, and the Tarski-Kantorovich Theorem

A set  $X$  is said to be **partially ordered**, or a **poset**, if it is nonempty and for certain pairs  $(x, y)$  in  $X \times X$  there is a binary relation  $x \leq y$  which satisfies:

1.  $x \leq x$  for each  $x \in X$ ;
2. if  $x \leq y$  and  $y \leq x$ , then  $x = y$ ;
3. if  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ .<sup>4</sup>

Suppose that  $Y \subseteq X$  and let  $X$  be a poset. The set  $Y$  is called a **chain** (of  $X$ ) if and only if  $Y$  is nonempty and for all  $x, y \in Y$ , one of the two conditions  $x \leq y$  or  $y \leq x$  holds. If the chain is countable, then it is called a **countable chain**. Let  $\{x^n\}_{n=0}^{\infty} \subset X$  be a monotone sequence. That is, either  $x^n \leq x^{n+1}$  for each  $n$ , or  $x^n \geq x^{n+1}$  for each  $n$ . A monotone sequence is a countable chain. The supremum and infimum of a monotone sequence are denoted in lattice notation as follows:

$$\bigvee x^n = \sup_n x^n; \text{ and } \bigwedge x^n = \inf_n x^n.$$

If, for every chain  $Y \subseteq X$ , we have  $\inf Y \equiv \bigwedge Y \in X$  and  $\sup Y \equiv \bigvee Y \in X$ , then  $X$  is said to be a **chain complete poset**. If this condition obtains only for every countable chain  $Y \subseteq X$ , then  $X$  is said to be a **countably chain complete poset**. The monotone sequence  $\{x^n\}_{n=0}^{\infty}$  is **increasing (decreasing)** when  $x^n \leq x^{n+1}$  ( $x^n \geq x^{n+1}$ ) for each  $n$ .

Let  $Y \subseteq X$  and let  $X$  once again be a poset. The element  $x \in Y$  is called **greatest (smallest)** in  $Y$  if and only if  $y \leq x$  ( $x \leq y$ ) respectively, for all  $y \in Y$ . If  $Y$  has greatest and smallest elements, then monotone sequences  $\{x^n\} \subseteq Y$  are countably chain complete posets in  $Y$ . A greatest element is also known as the **top** element; the smallest element is also known as the **bottom** element.

A poset  $X$  is a **lattice** provided each pair of elements has a supremum and an infimum. A **sublattice** of a lattice is a subset which is closed under pairwise infima and suprema. A **complete lattice** is a lattice in which each nonempty subset  $Y$  has a supremum  $\bigvee Y$  and an infimum  $\bigwedge Y$ . Note that a complete lattice has a top and a bottom element.

A function  $F : X \rightarrow X$  is said to be a **self-map on  $X$** . By  $F^N(x)$ , we are denoting the  $N^{\text{th}}$ -iteration of  $F$  with initial seed  $x$ . That is,  $F^N(x) = F(F^{N-1}(x))$  for each natural number  $N$  and  $F^0(x) \equiv x$ . This self-map is

<sup>3</sup>This links with work by Martins-da-Rocha and Vailakis [35] and [36].

<sup>4</sup>Birkhoff [11] covers the basic properties of posets and lattices.

said to be **monotone** whenever  $x, y \in X$  and  $x \leq y$ , then  $F(x) \leq F(y)$ . Some writers refer to a monotone self-map as an **isotone self-map or an increasing self-map**. A point  $x^* \in X$  with  $F(x^*) = x^*$  is a fixed point of the self-map,  $F$ . The set of all fixed points of this self-map is denoted  $\text{fix}(F)$ .

The self-map  $F$  has a **subsolution in  $X$**  provided  $\{x \in X : F(x) \geq x\}$  is nonempty. Likewise,  $F$  has a **supersolution in  $X$**  provided  $\{x \in X : F(x) \leq x\}$  is nonempty. Evidently  $\text{fix}(F)$  is the intersection of the set of subsolutions and supersolutions.

Suppose  $\langle \underline{x}, \bar{x} \rangle$  is an order interval in  $X$ ,  $\underline{x} \leq \bar{x}$  and  $\underline{x} \neq \bar{x}$ . That is,  $x \in \langle \underline{x}, \bar{x} \rangle$  if and only if  $\underline{x} \leq x \leq \bar{x}$ . Clearly  $\underline{x}$  is the smallest element of the order interval while  $\bar{x}$  is the corresponding greatest element. If  $F$  is a monotone self-map from  $\langle \underline{x}, \bar{x} \rangle$  to itself, then  $F$  has a subsolution,  $\underline{x}$ , and a supersolution,  $\bar{x}$ . Ziedler ([46], p. 69) cites a general principle that the presence of a subsolution and a supersolution implies the existence of a fixed point. Neither the subsolution nor the supersolution may be fixed points. For example, a self-map defined on a complete lattice has a subsolution, the bottom element, and a supersolution, the top element. However, neither the top nor bottom element may be a fixed point.

The classical Tarski Fixed Point Theorem [43] asserts that a monotone self-map on a complete lattice has a nonempty set of fixed points. Moreover, there is a smallest and a largest fixed point. These are the extremal fixed points. The set of all fixed points forms a complete sublattice of the given complete lattice. Successive approximations iterating the monotone self-map by transfinite induction yields the largest fixed point with initial seed the top element, and the smallest fixed point using the same procedure with initial seed the bottom element.<sup>5</sup> Iteration using transfinite induction is not a constructive procedure in any sense of that term.

The Tarski-Kantorovich Theorem is similar to Tarski's result, but built on weaker properties for the monotone self-map's domain and a stronger condition on that operator in the form of an order continuity requirement. There are extremal fixed points. This theorem also illustrates Ziedler's principle when the set  $X$  in question is an order interval in an appropriate ordered vector space, and the self-map is monotone and exhibits a form of order continuity. The order interval has both a top and bottom element which serve as the supersolution and subsolution, respectively. The specific form of the continuity property is introduced below and also implies the set of fixed points is a countably chain complete subset of the operator's domain. We follow the approach taken by Balbus, Reffett and Woźny [6]. The successive approximation procedure used in this result is constructive in so far as the iterations are indexed on the natural numbers in contrast to the iterative procedure underlying Tarski's Theorem.

**Definition 1** *A self-map  $F$  defined on a countably chain complete poset  $X$  with the greatest element  $\bar{x}$  and smallest element  $\underline{x}$  is **monotonically sup-***

<sup>5</sup>See Cousot and Cousot [14] and Echenique [17] for this transfinite induction argument.

**preserving** if for any increasing  $\{x^n\}$  we have

$$F\left(\bigvee x^n\right) = \bigvee F(x^n),$$

and **monotonically inf-preserving** if for any decreasing  $\{x^n\}$ , we have

$$F\left(\bigwedge x^n\right) = \bigwedge F(x^n).$$

$F$  is said to be **monotonically sup-inf-preserving** if and only if it is both monotonically sup-preserving and monotonically inf-preserving.

Evidently, a monotonically sup (respectively, inf)-preserving self map on the ordered space  $X$  must be an increasing self-map. The sup-inf preservation property is a type of **order continuity**.<sup>6</sup> In the case of a monotonically increasing sequence the sup is regarded as the sequence's limit and continuity is taken to mean  $F(\sup\{x_n\}) = \sup\{F(x_n)\}$  where the countable chain is denoted  $\{x_n\}$ . Likewise for the inf of a decreasing sequence. Some authors (e.g. Granas and Dugundji [18]) refer to order continuity as used here by the term  $\sigma$  - **order continuity** to stress the restriction to countable chains and also drop the monotonicity requirement for the sequences.

The Tarski-Kantorovich Theorem (Granas and Dugundji [18], p. 26) as refined by Balbus, Reffett and Woźny ([6], Theorem 7) states the following:<sup>7</sup>

**Theorem 2 Tarski-Kantorovich.** *Suppose that  $X$  is a countably chain complete partially ordered set with the greatest element,  $\bar{x}$ , and the smallest element,  $\underline{x}$ . Let  $F$  be a monotone self-map on  $X$ .*

1. *If  $F$  is monotonically inf-preserving; then  $\bigwedge F^N(\bar{x})$  is the greatest fixed point of  $F$ , denoted  $x^\infty$ ;*
2. *if  $F$  is monotonically sup-preserving; then  $\bigvee F^N(\underline{x})$  is the smallest fixed point of  $F$ , denoted  $x_\infty$ .*
3.  *$\text{fix}(F) = \langle x_\infty, x^\infty \rangle$  is an order an order interval in  $X$ . It is also a nonempty countably chain complete poset in  $X$ . Moreover,*

$$\begin{aligned} x^\infty &= \bigvee \{x \in X : F(x) \geq x\}; \\ x_\infty &= \bigwedge \{x \in X : F(x) \leq x\}. \end{aligned}$$

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<sup>6</sup>This notion of order continuity is identical to continuity of  $F$  in the Scott topology in our application spaces. There is a different notion of order continuity in the Riesz space literature that differs materially from continuity of our self-map in the Scott topology. See Aliprantis and Border [1] for the Riesz space version of order continuity. Becker and Boyd [9] link the Riesz space version to an interpretation to myopia in the context of consumer preferences.

<sup>7</sup>Baranga [8] presents the "Kleene Fixed Point Theorem." Jachymski et al ([21], p. 249) argues this result is equivalent to the Tarski-Kantorovich Theorem as quoted in our text. Also, see Stoltenberg-Hansen, et al ([44], p. 21) on Kleene's Fixed Point Theorem. Kamihigashi et al [22] apply the Kleene Fixed Point Theorem to dynamic programming.

The result that  $\text{fix}(F)$  is a countably chain complete poset in  $X$  is due to Balbus, Reffett, and Woźny [6]. It is the analog of Tarski's result that  $\text{fix}(F)$  is a complete lattice in his setting. The Tarski-Kantorovich theorem tells us that successive approximations (iteration of  $F$  indexed on the natural numbers) initiated at either the smallest or greatest element of the set  $X$  produces the smallest or greatest fixed point in the limit, respectively.<sup>8</sup> Moreover, it is clear that  $x_\infty \leq x^\infty$ . If  $x^*$  is any other fixed point for  $F$ , then  $x_\infty \leq x^* \leq x^\infty$ . The fixed point  $x_\infty$  is the smallest (infimum) of the set of supersolutions while  $x^\infty$  is the largest (supremum) of the set of subsolution to the operator equation  $F(x) = x$ .

## 2.2 Positive Cones and Nonlinear Operators in Real Banach Spaces

Let  $E$  denote a real Banach space. The zero element in  $E$  is denoted by  $\theta$ . A nonempty norm-closed convex subset  $P$  of  $E$  is said to be a **cone** if  $x \in P$ , then  $\lambda x \in P$  for each scalar  $\lambda \geq 0$ .<sup>9</sup> In particular this definition of a cone implies  $\theta \in P$ . A cone is used to induce a partial order on the vectors belonging to  $E$ . A vector  $x$  is said to be **positive**, written  $x \geq \theta$ , provided  $x \in P$ . The cone is then called the **positive cone** of  $E$ . The standard partial relation expressing  $x \geq y$  whenever  $x, y \in E$  is defined by requiring  $x - y \in P$ . Write  $x > \theta$  whenever  $x \geq \theta$  and  $x \neq \theta$ . Likewise,  $x > y$  provided  $x \geq y$  and  $x \neq y$ . The notation  $x \gg \theta$  means  $x \in \text{int}(P)$ , where  $\text{int}(P)$  denotes the interior of the cone  $P$ . Of course, this latter inequality is only meaningful when  $\text{int}(P) \neq \emptyset$  — a strong topological restriction on the underlying Banach space. A cone  $P$  with nonempty interior in its norm topology is said to be a **solid cone**. We assume without further mention in this paper that  $E$  is equipped with the partial order derived from the cone  $P$ . The notation for the positive cone of the underlying space,  $E$ , is also denoted by  $E^+$ .

We consider an abstract nonlinear operator, denoted by  $A$ , that is positive on this cone. That is, it is a self-map on this cone:  $A : P \rightarrow P$ . We write this as the requirement  $AP \subseteq P$ . The term nonlinear as used here formally includes the possibility that the operator is linear. The Koopmans operator is a nonlinear operator which is positive on a cone in a particular Banach lattice. However, it is not a linear operator on that set.

The operator  $A : P \rightarrow P$  is said to be **monotone (isotone, increasing) on  $P$**  if  $x \leq y, (x, y \in P)$  implies  $Ax \leq Ay$ . The Koopmans operator is **monotone** whenever the aggregator is also monotone in its arguments.

Given an abstract nonlinear operator satisfying  $AP \subseteq P$  we are concerned with the *existence of fixed points* as well as whether or not there is a *unique*

<sup>8</sup>This conclusion can fail without order continuity. See Davey and Priestley ([15], p.93) for a counterexample.

<sup>9</sup>This definition is not used by all authors (see Aliprantis and Tourky [4]). We follow the usage in Krasnosel'skii's books referenced below. In particular, we follow the conventions in [29]. In our application  $P$  is defined in terms of a given partial order on a function space. Additional properties, such as  $P$  is norm closed, must be verified; likewise  $P$  is convex.



solution in the cone  $P$ . The **operator equation** is  $Ax = x$  with  $x \in P$ ; a solution is a fixed point of the operator,  $A$ . That is, a point  $x \in P$  such that  $Ax = x$ . In some applications there may be a trivial fixed point,  $\theta$ . We are primarily interested in **nontrivial fixed points**  $x \in P$  with  $x \neq \theta$ . The Koopmans operator does not admit a trivial fixed point under our assumptions.

The spaces we encounter in this working paper are complete normed Riesz spaces. They are also Banach lattices. See Aliprantis and Border [1], Aliprantis and Burkinshaw [2], Meyer-Nieberg [38], Peressini [39], and Vulikh [45] for details on Riesz spaces and Banach lattices. Indeed, the spaces on which the Koopmans operator acts turn out to be *abstract  $M$ -spaces, or  $AM$ -spaces with an order unit*. One advantage to this setup is that the positive cones are norm-closed, convex sets and are solid — they possess nonempty norm interiors.<sup>10</sup>

### 3 Recursive Utility Theory for The Thompson Aggregator Class

We review the Marinacci and Montrucchio [37] recovery theorem that yields extremal solutions to the Koopmans equation belonging to a Thompson aggregator. We begin with the defining properties of these aggregators. Our version of the their recovery theorem clarifies the order continuity property enjoyed by the Koopmans operator. This property is critical for iteration of the Koopmans operator from well-chosen initial seeds to yield extremal fixed points via an iterative procedure as promised in their paper. The added advantage of this clarity is the insure the extremal fixed points possess semicontinuity properties: the smallest fixed point is a lower semicontinuous function on the underlying commodity space and the largest fixed point is an upper semicontinuous fixed point.

#### 3.1 Defining Properties of Thomson Aggregators

The class of Thompson aggregators is delineated by the following four basic assumptions.

**Definition 3**  $W : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  is said to be a **Thompson aggregator** if it satisfies properties (T1) – (T4):

- (T1)  $W \geq 0$ , continuous, and monotone:  $(x, y) \leq (x', y')$  implies  $W(x, y) \leq W(x', y')$ ;
- (T2)  $W(x, y) = y$  has at least one nonnegative solution for each  $x \geq 0$ ;

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<sup>10</sup>The positive cone of an AM-space has a nonempty norm interior provided it is an AM-space with unit. See Aliprantis and Tourky ([4], p. 64) and Peressini ([39], p. 183). A Banach lattice has an order unit if and only if that order unit is an interior point of the space's positive cone. In this case, the original sup norm and lattice norm topologies are equivalent. See Meyer-Nieberg ([38], Corollary 1.2.14 for details).

(T3)  $W(x, \bullet)$  is concave at 0 for each  $x \geq 0$ , that is

$$W(x, \mu y) \geq \mu W(x, y) + (1 - \mu) W(x, 0)$$

for each  $\mu \in [0, 1]$  and each  $(x, y) \in \mathbb{R}_+^2$ ;

(T4)  $W(x, 0) > 0$  for each  $x > 0$ .

Our definition of a Thompson aggregator builds in the assumption that it is jointly continuous in  $(x, y)$  over  $\mathbb{R}_+^2$ . Marinacci and Montrucchio [37] prove that their definition of a Thompson aggregator is jointly continuous with  $y$  restricted to the open interval  $(0, \infty)$ . For technical reasons we require joint continuity as well as continuity at  $y = 0$  whatever value is assumed by  $x$ . They admit this in the formal assumptions for their recovery theorem. We prefer to build this joint continuity assumption directly into the definition of a Thompson aggregator as the known examples satisfy it. Moreover, it is this property that plays critical roles in the existence theory. In particular, it is needed to prove the Koopmans operator belonging to a Thompson aggregator maps a suitably bounded continuous function into another suitably bounded continuous function. Moreover, this assumption is critical to the verification that the Koopmans operator enjoys the order continuity property required for the Tarski-Kantorovich Theorem's application. This condition also shows up in our demonstration that there is an upper semicontinuous (lower semicontinuous) extremal solution to the Koopmans equation using the Tarski-Kantorovich Theorem. For these reasons our existence argument differ from the parallel one given by Marinacci and Montrucchio built on Tarski's Fixed Point Theorem [43]. In addition, they impose two additional properties formalized here as assumptions (T5) and (T6).<sup>11</sup> These conditions are essential ingredients to the proof of their recovery theorem. Both properties further restrict the class of Thompson aggregators from which an underlying recursive utility representation is possible.

The first additional condition imposed by Marinacci and Montrucchio is the aggregator be  $\gamma$ - **subhomogeneous**.

(T5)  $W$  is  $\gamma$ - subhomogeneous — there is some  $\gamma > 0$  such that:

$$W(\mu^\gamma x, \mu y) \geq \mu W(x, y)$$

for each  $\mu \in (0, 1]$  and each  $(x, y) \in \mathbb{R}_+^2$ .

The *standard positive homogeneity* of degree  $\gamma$  aggregator functional form corresponds to the case where

$$W(\mu x, \mu y) = \mu^\gamma W(x, y)$$

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<sup>11</sup>Their limit condition is assumed in their results, but it is not listed as a separate axiom. They also study the role of this condition in the context of Blackwell aggregators (see Lemma 2 in Appendix D).

for each  $\mu > 0$  and each  $(x, y) \in \mathbb{R}_+^2$ . If the defining inequality in (T5) is an equality, then we say  $W$  is  $\gamma$ -homogeneous.<sup>12</sup> We turn to the second property required for the recovery theorem's proof.

(T6) *W satisfies the MM-Limit Condition: for a given  $\alpha \geq 1$  and  $\gamma > 0$  (from (T5)),*

$$\lim_{t \rightarrow \infty} \frac{W(1, t)}{t} < \alpha^{-1/\gamma}, \quad (1)$$

with  $t > 0$ .

The parameter  $\alpha$  in (T6) is the economy's maximum possible consumption growth factor in applications. Condition (T6) turns out to be an important **joint restriction** on the preferences embodied in the aggregator function as well as on the underlying commodity space, as might arise from properties of technologies in production economies and/or endowments in exchange economies.<sup>13</sup> Condition (1) may not obtain for an arbitrarily chosen member of the Thompson class given the  $\alpha$  parameter's value. Joint restrictions of this type routinely appear in treatments of the Blackwell aggregator class. What is certainly true under assumptions (T1) – (T5) is that

$$L \equiv \lim_{y \rightarrow \infty} \frac{W(1, y)}{y} \quad (2)$$

exists as the ratio  $W(x, y)/y$  is decreasing in  $y$  and bounded below by zero as formally demonstrated by Marinacci and Montrucchio [37]. But, this limit,  $L$ , could be larger or smaller than  $\alpha^{-1/\gamma}$ . Certainly if  $L = 0$ , then (1) holds. We list satisfaction of the MM Limit Condition as an explicit axiom that might, or might not, obtain for a particular aggregator in order to emphasize that some restrictions may apply on the underlying model's deep preference and technology parameters. Restrictions (if any) on preferences and technology implicit in (1) are illustrated in the following examples.

### 3.2 Examples of Thompson Aggregators

There are two important sources for examples. The KDW aggregator (defined below) has some parameterization which take it outside the Blackwell class and place it firmly in the Thompson family. There are also many new examples based on the Constant Elasticity of Substitution functional form for utility functions and production functions commonly studied in microeconomic theory. Both the CES and KDW examples illustrate the finer properties of Thompson aggregators that are also required to meet (T5) and (T6).

<sup>12</sup>Dolmas [16] introduces a class of  $\gamma$ -homogeneous aggregators in order to demonstrate the existence of a balanced growth path in suitable optimal growth models with planner's utility specified beyond the conventional time additive discounted utility case. His definition of a  $\gamma$ -homogeneous aggregator differs however: he requires  $W(\alpha x, \alpha^\gamma y) = \alpha^\gamma W(x, y)$ .

<sup>13</sup>Becker and Boyd [9] cover many examples of models with commodity spaces arising in intertemporal choice and consistent with our Thompson aggregator specification.

### 3.2.1 CES Aggregators

Standard utility theory for two, or more, goods suggests the CES class as a potential source for aggregators. Certainly, CES utility functions over two dated consumption goods, one good corresponding to today's consumption, and the other to tomorrow's consumption, are reasonable and widely applied in equilibrium theory. Indeed, these forms are often taken as the standard specifications! The Fisherian inspired reinterpretation of the aggregator's second argument as *future utility*, is the economic basis for our interest in aggregator models! This suggests introducing the corresponding class of **CES aggregators** defined by the formula:

$$W(x, y) = (1 - \beta)x^\rho + \beta y^\rho, \text{ for } 0 < \rho \leq 1. \quad (3)$$

The parameter  $\beta$  is restricted —  $0 < \beta < 1$ . Note that this family of functions is positively homogeneous of degree  $\rho$ .<sup>14</sup>

The **elasticity of substitution** is the parameter

$$\sigma := \frac{1}{1 - \rho}; \rho \neq 1.$$

The case where  $\rho = 1$  is equivalent to specifying  $W$  as a linear function of  $x$  and  $y$ . In that case, consumption today,  $x$ , and future utility,  $y$ , are “perfect” substitutes with an infinite elasticity of substitution. The case  $\rho = 0$  corresponds to the *Cobb-Douglas aggregator* where  $\sigma = 1$ .<sup>15</sup> The restriction  $0 < \rho < 1$  otherwise imposed is required to insure  $W$  is both a homothetic and concave function in the variables  $(x, y) \in \mathbb{R}_+^2$  with  $W(x, y) \geq 0$  and  $W(0, 0) = 0$ . These aggregator functions are unbounded from above. This is an important point for developing an appropriate recovery theorem. Verification of property (T3) also follows from the fact  $W$  is jointly concave in  $(x, y)$ , a fact that may NOT be true for an arbitrary Thompson aggregator. This joint concavity condition plays a critical role in our uniqueness theorem. The other Thompson aggregator criteria are met when  $\sigma > 1$ . *We work exclusively with specifications satisfying  $\sigma > 1$  for the remainder of our paper.* The Cobb-Douglas case,  $W(x, y) = x^{1-\beta}y^\beta$  for  $0 < \beta < 1$ , fails (T4). Hence, it is not a Thompson aggregator. Marinacci and Montrucchio [37] show how to modify the basic Thompson axioms and derive a suitable solution to the Koopmans equation for this particular CES aggregator. The next issue is to certify that our CES Thompson aggregators satisfy (T5) and (T6).

Property (T5) holds for the CES aggregator family. They are  $\gamma$ -subhomogeneous with  $\gamma = 1$ . Indeed, the CES aggregators are positively homogeneous of degree  $\rho$ . Formally, for  $\gamma = 1$  and  $0 < \rho < 1$ ,

$$\begin{aligned} W(\mu x, \mu y) &= (\mu^\rho) [(1 - \beta)x^\rho + \beta y^\rho] \\ &\geq \mu W(x, y) \text{ as } \mu^\rho \geq \mu \text{ since } 0 < \mu \leq 1. \end{aligned}$$

<sup>14</sup>See the Appendix for a discussion of Koopmans equivalent aggregator formulations that preserve ordinal utility properties.

<sup>15</sup>Strictly speaking, this observation results when the parameter  $\rho \rightarrow 0$  for the aggregator  $V(x, y) = [(1 - \beta)x^\rho + \beta y^\rho]^{1/\rho}$ . See the Appendix for additional commentary on the aggregator  $V$ . Of course both  $\beta$  and  $\rho$  belong to the interval  $(0, 1)$  in this aggregator specification.

The CES aggregators satisfy the MM-Limit Condition. Consider the case  $\rho = 1/2$ . Form the average at  $x = 1$  and take the limit as  $y \rightarrow \infty$  :

$$\lim_{y \rightarrow \infty} \frac{W(1, y)}{y} = \lim_{y \rightarrow \infty} \frac{(1 - \beta) + \beta\sqrt{y}}{y} = 0.$$

Routine calculations show that for the CES aggregator  $W(x, \bullet)$  does **not** satisfy a Lipschitz condition in  $y \geq 0$  whenever  $0 < \rho < 1$ . Just compute  $W_2 \equiv \partial W / \partial y$  and note  $\sup_{y \geq 0} W_2(x, y) = +\infty$ . This aggregator specification fails to exhibit the discounting property qualifying it for Blackwell aggregator status.<sup>16</sup> Suppose, for example, that  $\rho = 1/2$ . Then

$$W_2 = \frac{\beta}{2} (y)^{-\frac{3}{2}} \rightarrow +\infty \text{ as } y \rightarrow 0^+.$$

Hence,  $\sup_{y \geq 0} W_2 = +\infty$ . This CES aggregator is **not** a Blackwell aggregator. Of course, if  $\rho = 1$ , then the corresponding CES aggregator is also a member of the Blackwell aggregator family. In this situation,  $0 < \beta < 1$  plays the role of the utility discount factor.

Marinacci and Montrucchio introduce a four parameter family of aggregators which are variants of the CES class: set

$$W(x, y) = (x^\eta + \beta y^\xi)^{1/\rho}, \quad (4)$$

where  $\eta, \xi, \rho, \beta > 0$ . Conditions (T1) and (T4) always hold. If  $\xi \leq 1$ , then this aggregator IS a Thompson aggregator in two cases:

- (i)  $\xi < \rho$ , or
- (ii)  $\xi = \rho$  and  $\beta < 1$ .

Property (T5) holds with  $\gamma = \xi/\eta$ , provided  $\xi \leq \rho$ . In this case, the aggregator is  $\gamma$ -subhomogeneous. Property (T3) follows provided  $\xi \leq 1$  and  $\xi \leq \rho$ . Notice that this aggregator is jointly concave provided  $\rho < 1$  and  $\xi < 1$  as well.

They also discuss the special case where  $\eta = 1, 0 < \beta < 1, \xi = \rho$ , and  $0 < \rho < 1$ . Here,  $W(x, y) = (x + \beta y^\rho)^{1/\rho}$ . This aggregator is  $\rho$ -homogeneous:

$$W(\alpha^\rho x, \alpha y) = \alpha W(x, y) \text{ for each } \alpha \geq 0.$$

We note that this aggregator is also Koopmans equivalent to the TAS aggregator  $W^*(x, y) = x + \beta y$ , a member of the Blackwell aggregator family.<sup>17</sup>

<sup>16</sup>Recall, this is the requirement  $0 < \sup_{y \geq 0} W_2(x, y) < 1$  for differentiable aggregators such as the examples developed here.

<sup>17</sup>See the discussion in Becker and Boyd ([9], p. 74).

### 3.2.2 KDW Aggregators

Koopmans, Diamond, and Williamson [26] introduced an interesting aggregator. We refer to it as the **KDW aggregator**. It is defined by the formula

$$W(x, y) = \frac{\delta}{d} \ln(1 + ax^b + dy)$$

where  $a, b, d, \delta > 0$ .

The KDW aggregator fails to satisfy the required Blackwell contraction condition in the event the parameter  $\delta \geq 1$ . Recall this aggregator always satisfies a Lipschitz condition in its second argument. Fix the value of  $x \geq 0$ . It is easy to verify that the partial derivative of  $W$  with respect to  $y$ , denoted  $W_2$ , is given by

$$W_2 = \frac{\delta}{(1 + ax^b + dy)}.$$

Clearly  $(1 + ax^b + dy) \geq 1$  for all  $(x, y) \geq 0$ . Thus,

$$W_2(x, y) \leq \delta.$$

This inequality is valid for any  $\delta > 0$ . This proves

$$\sup_{y \geq 0} W_2(x, y) \leq \delta,$$

and this aggregator always satisfies a Lipschitz condition in its second argument. The Blackwell aggregator case occurs when  $\delta < 1$ . Suppose  $\delta \geq 1$ . Let  $\delta = 1$  first. Then

$$\sup_{y \geq 0} W_2(x, y) \leq 1.$$

Now consider the special case when  $x = 0$  and observe:

$$\sup_{y \geq 0} W_2(0, y) = \sup_{y \geq 0} \frac{1}{(1 + dy)} = 1$$

since  $1 + dy \rightarrow 1$  as  $y \rightarrow 0^+$ . The uniform contraction property for Blackwell aggregators fails even though this aggregator does satisfy a Lipschitz condition in  $y$ . Likewise, we cannot conclude the uniform contraction property obtains whenever  $\delta > 1$ . Just notice

$$\sup_{y \geq 0} W_2(0, y) = \sup_{y \geq 0} \frac{\delta}{(1 + dy)} = \delta > 1.$$

The KDW aggregator satisfies  $\gamma$ -subhomogeneity for  $\gamma = b^{-1}$ . That is, (T5) holds for the KDW aggregator. It is interesting to note that (T5) applies to both current consumption and future utility arguments, whereas the question of discounting or not is a property of the future utility argument alone as well as parameter  $\delta$ 's magnitude.

The verification the KDW is  $b^{-1}$ -subhomogeneous proceeds as follows. Let  $z \geq 0$  be a fixed number and define the function  $g$  for each  $\mu \in (0, 1]$  according to the formula:

$$g(\mu) = \ln(1 + \mu z) - \mu \ln(1 + z).$$

Actually, we can also extend this definition to the value  $\mu = 0$  since  $g$  is well-defined at 0 and is, in fact, continuous at 0. Note that  $g(0) = 0$  and  $g(1) = 0$  too. Moreover,

$$g''(\mu) = -\frac{z^2}{(1 + \mu z)^2} < 0 \text{ for } \mu \in (0, 1).$$

Hence,  $g$  is (strictly) concave on  $[0, 1]$ . Clearly  $g(\mu) > 0$  for each  $\mu \in (0, 1)$ . Thus, with  $z = ax^b + dy$  we have for each  $\mu \in (0, 1]$ :

$$\begin{aligned} W(\mu^{b^{-1}}x, \mu y) &= \frac{\delta}{d} \ln\left(1 + a(\mu^{b^{-1}}x)^b + d(\mu y)\right) \\ &= \frac{\delta}{d} \ln(1 + \mu(ax^b + dy)) \\ &\geq \mu \frac{\delta}{d} \ln(1 + ax^b + dy) \text{ (as } \mu \in (0, 1]) \\ &= \mu W(x, y). \end{aligned}$$

The KDW aggregator is an example of a  $\gamma$ -subhomogeneous (with  $\gamma = b^{-1}$ ) aggregator that is NOT a homogeneous aggregator, like members in the CES family. This example also illustrates why (T5) only requires  $\gamma > 0$ . IF the parameter  $0 < b < 1$  (so the KDW aggregator is concave in  $x$  for each  $y$ ), then  $\gamma > 1$  in checking that (T5) obtains. By way of contrast, the parameter  $\gamma = 1$  for  $W$ -class CES Thompson aggregators, their Koopmans equivalent versions, and the  $V$ -class CES aggregators.

The KDW aggregator also satisfies (T6). That is, the limit  $L = 0$  in (1). Here, just notice for  $x = 1$ ,

$$\frac{W(1, y)}{y} = \frac{\ln(1 + a + dy)}{y} \rightarrow 0 \text{ as } y \rightarrow \infty$$

for **any**  $a, d \geq 0$ . In this case, (T6) holds for **any**  $a \geq 1$ .

## 4 Recovery Theory: Marinacci and Montrucchio's Theorem

### 4.1 The Setup

Marinacci and Montrucchio [37] prove a Recovery Theorem for Thompson aggregators whenever the underlying commodity space is a principal ideal of the vector space of all real-valued sequences,  $s$ , with the usual coordinatewise partial order. Given a non-zero vector  $\omega \in s^+$ , the set

$$A_\omega = \{X \in s : |X| \leq \lambda \omega \text{ for some scalar } \lambda > 0\}$$

defines a **principal ideal** in  $s$ . Use the notation  $0$  for the zero vector of this commodity space and reserve  $\theta$  for the real-valued zero function,  $\theta(X) = 0$ , defined on this space. The positive cone of  $A_\omega$  is:

$$A_\omega^+ = \{X \in A_\omega : X \geq 0\}.$$

It serves as the commodity space in the anticipated economic applications.<sup>18</sup>

We generally consider two cases of this commodity space on economic grounds: the first occurs when  $\omega = (1, 1, \dots)$ , and  $A_\omega = \ell_\infty$ , the vector space of all bounded real-valued sequences. The second case arises in the general exponential model where  $\omega = (\alpha, \alpha^2, \dots)$  for  $\alpha \geq 1$ . In the latter situation we recall  $\ell_\infty \subset A_\omega \subset s$  when  $\alpha > 1$ . Our version of Marinacci's and Montrucchio's recovery theory applies to exponential models where  $\alpha \geq 1$ . Thus, we always assume the vector  $\omega$  is strictly positive in each component. This implies  $\omega$  is an **order unit** in the space  $A_\omega$ . Furthermore, for each  $x \in A_\omega$ ,

$$\|x\|_\infty = \inf \{\lambda > 0 : |x| \leq \lambda\omega\}$$

defines a *lattice norm* on that space. Here,  $\lambda$  is a scalar. A norm  $\|\bullet\|$  on a Riesz space is a **lattice norm** whenever  $|x| \leq |y|$  implies  $\|x\| \leq \|y\|$  where  $|x|$  is the absolute value of the vector  $x$ . The absolute value of a point in  $A_\omega$  turns out to be the absolute value taken component-by-component:  $|x| = \{|x_t|\}_{t=1}^\infty$ . This lattice norm is topologically equivalent to the standard sup *-norm* developed below.

Following ideas drawn from Boyd [12], and further developed in Becker and Boyd [9], weighted norms are introduced on this principal ideal. These norms are deduced using strictly positive real-valued weight functions defined on  $A_\omega$ . These weight functions are expressed in particular functional forms in aggregator models. These functions are specifically chosen to be well-adapted to the application at hand. There are two distinct uses of weight functions. First, we use the lattice norm inherited from the given principal ideal to define a weighted norm on the set  $A_\omega$  that turns it into a Banach space in its own right. Second, we introduce another weight function to form a space of bounded functions according to this weight function. These functions are real-valued and defined on  $A_\omega^+$ , the positive cone of  $A_\omega$ . Think of these functions as possible trial utility functions on the underlying commodity space. We seek a solution to the Koopmans operator equation in this function space. Marinacci and Montrucchio's weight function is chosen in this latter case in order to construct a particular order interval of trial functions on which the solution to Koopmans' equation is sought.

The  $\alpha$ -*norm*,  $\|\bullet\|_\alpha$ , is defined for elements of  $A_\omega$  by the formula:

$$\|C\|_\alpha = \sup_{t \geq 1} \left| \frac{c_t}{\alpha^t} \right|. \quad (5)$$

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<sup>18</sup>Becker and Boyd [9] illustrate a range of applications and present arguments for postulating this positive cone as the commodity space.



The normed vector space  $\ell_\infty(\alpha)$  is defined by the pair  $(A_\omega, \|\bullet\|_\alpha)$  where  $\alpha \geq 1$ . We note that the sequences in this space are  $\alpha$ -norm-bounded since  $(|c_t|/\alpha^t) \leq \lambda < +\infty$ . This is so as  $C \in A_\omega$  means there is some scalar  $\lambda > 0$  such that  $|c_t| \leq \lambda\alpha^t$  for each  $t$ . Hence,  $\|C\|_\alpha \leq \lambda < +\infty$  whenever  $C \in A_\omega$ . This normed space is a vector lattice with the usual pointwise operations for join and meet of two vectors. The positive cone of this space is denoted by  $\ell_\infty^+(\alpha)$ , which is just  $A_\omega^+$  with the relative  $\alpha$ -norm topology. The space  $\ell_\infty(\alpha)$  is also a Banach lattice, so its positive cone is also  $\alpha$ -norm closed. The lattice norm is equivalent to the  $\alpha$ -norm. This positive cone is also convex and has a nonempty  $\alpha$ -norm interior. The latter fact follows from the observation that  $\ell_\infty(\alpha)$  is an AM-space with unit vector  $\omega$ .

We turn to the second weight function. We need to define a set of possible, or trial, real-valued utility functions with common domain  $\ell_\infty^+(\alpha)$ . These trial utility functions must also be bounded in an appropriately defined norm. The next weight function enters at this stage in order to define a suitable space of “bounded” real-valued functions on the commodity space.

First, define a **weight function**,  $\varphi_\gamma$  following Marinacci and Montrucchio’s [37] specification. For each  $C \in \ell_\infty(\alpha)$  define  $\varphi_\gamma$  by the formula:

$$\varphi_\gamma(C) = (1 + \|C\|_\alpha)^{1/\gamma}. \quad (6)$$

This weight function is uniformly continuous on  $\ell_\infty^+(\alpha)$  with respect to the  $\alpha$ -norm topology. Here, the parameter  $\gamma > 0$  appearing in the weight function is taken from (T5). This weight function as well the  $\alpha$ -norm entangle preference and technology parameters — the growth rate  $\alpha$  is derived from a model’s technology side while the parameter  $\gamma$  comes from the model’s preference side.

**Definition 4** A function  $U : \ell_\infty^+(\alpha) \rightarrow \mathbb{R}$  is  $\varphi_\gamma$ -bounded provided

$$\|U\|_\gamma := \sup_{C \in \ell_\infty^+(\alpha)} \frac{|U(C)|}{(1 + \|C\|_\alpha)^{1/\gamma}} < +\infty.$$

The set of all  $\varphi_\gamma$ -bounded real-valued functions with domain  $\ell_\infty^+(\alpha)$  is denoted by  $F_\gamma^\alpha$ .

The **zero function**,  $\theta$ , is defined by  $\theta(C) = 0$  for each  $C$ . The zero function is the origin in the vector space  $F_\gamma^\alpha$ .

The space

$$C_\gamma^\alpha := \{U \in F_\gamma^\alpha : U \text{ is } \|\bullet\|_\infty\text{-continuous on } \ell_\infty^+(\alpha)\}$$

is a closed subspace of  $F_\gamma^\alpha$ . However, this space is **not** a complete lattice. The corresponding positive cone, denoted  $(C_\gamma^\alpha)^+$ , is a solid cone. Its weighted sup norm interior is nonempty since the weight function  $\varphi_\gamma \in (C_\gamma^\alpha)^+$  is an order unit.<sup>19</sup> This property is important for the Recovery Theorem and the conclusion

<sup>19</sup>The norm  $\|\bullet\|_\infty$  is a uniformly continuous real-valued function defined on the set  $A_\omega$ . See Aliprantis and Burkinshaw ([3], p. 218). Hence, the function  $\varphi_\gamma(C)$  is continuous as the composition of the continuous functions  $1 + \|C\|_\alpha$  and  $\phi(x) = x^{1/\gamma}$  for  $x > 0$ .

that the greatest fixed point of the Koopmans operator is a sup norm upper semicontinuous  $\varphi_\gamma$ -bounded real-valued function on  $\ell_\infty^+(\alpha)$ .

The aggregator approach to recovering recursive utility representations of an underlying preference relation defined on the given commodity space is expressed in terms of a functional equation. This equation takes the aggregator function as the primitive concept. The **Koopmans equation for recursive utility** is

$$U(C) = W(c_1, U(SC)). \quad (7)$$

Define the **shift operator**  $S : \ell_\infty^+(\alpha) \rightarrow \ell_\infty^+(\alpha)$  according to the rule  $C = \{c_1, c_2, c_3, \dots\} \mapsto SC = \{c_2, c_3, \dots\}$ . A solution of this equation is a recursive utility function representation of the preference relation. Of course, it all depends on what is meant by a solution. Proving this functional equation has a solution turns on recasting the problem as demonstrating a corresponding non-linear operator, known as the **Koopmans operator** (denoted by  $T_W$ ) has a fixed point in the desired function space of possible solutions. The Koopmans operator (defined below) is formally defined given a function  $U \in (F_\gamma^\alpha)^+$  by the following equation for each  $C \in \ell_\infty^+(\alpha)$ :

$$(T_W U)(C) = W(c_1, U(SC)).$$

If  $T_W U = U$ , then  $U$  is a solution to the Koopmans equation and defines a recursive representation of the underlying preference relation.

The Koopmans operator enjoys a monotonicity property whenever the aggregator is specified by a member of the Thompson aggregator class.

**Lemma 5** *If  $W$  is a Thompson aggregator, then  $T_W : (F_\gamma^\alpha)^+ \rightarrow (F_\gamma^\alpha)^+$  is a monotone operator.*

**Proof.** Suppose  $U, U' \in (F_\gamma^\alpha)^+$  and  $U \geq U'$ . Then  $T_W U \geq T_W U'$  since for each  $C \in \ell_\infty^+(\alpha)$  we have  $T_W U(C) = W(c_1, U(SC)) \geq W(c_1, U'(SC)) = T_W U'(C)$  as  $W$  is monotone increasing according to (T1). ■

A fixed point of the Koopmans operator belongs to  $(F_\gamma^\alpha)^+$ . *It may fail to possess any useful analytically or economically important properties.* For example, this fixed point may not be continuous, or even upper semicontinuous. Hence, *we seek at least one solution with mathematical properties appropriate for analyzing an intertemporal choice model.* Showing the model has an optimal solution is the first step in this analysis. Some form of continuity for the objective function is usually required to demonstrate an optimum exists.

Our objective is to show the Koopmans equation (7) has at least one economically interesting solution in the space  $(F_\gamma^\alpha)^+$  using the monotonicity property of the Koopmans operator when the aggregator belongs to the Thompson class, is  $\gamma$ -subhomogeneous and satisfies the MM Limit Condition. By an economically interesting solution we mean one that enjoys some form of continuity property. In fact, we show there is an upper semicontinuous solution in the space  $(F_\gamma^\alpha)^+$  and this is the largest fixed point in  $(F_\gamma^\alpha)^+$ . The formal statement of these facts is the **Marinacci and Montrucchio [37] Recovery Theorem**.

**Theorem 6** (Marinacci and Montrucchio [37]). *Suppose  $W$  is a Thompson aggregator satisfying (T5) and (T6). Then there is a  $\|\bullet\|_\alpha$  – upper semicontinuous function  $U^\infty \in (F_\gamma^\alpha)^+$  such that  $T_W U^\infty = U^\infty$ .*

Our proof of this Recovery Theorem is based on verifying the hypotheses of the Tarski-Kantorovich Theorem are met on an appropriately chosen order interval in the positive cone  $(F_\gamma^\alpha)^+$ . This order interval is denoted  $\langle \theta, U^T \rangle$ , where  $\theta$  is the zero function in  $(F_\gamma^\alpha)^+$  and  $U^T$ , called “ $U$ -top,” is defined below.

Of course we require  $U^T \in (F_\gamma^\alpha)^+$  as well. The desired order interval has the property  $T_W \theta \geq \theta$  and  $T_W U^T \leq U^T$  with  $T_W : \langle \theta, U^T \rangle \rightarrow \langle \theta, U^T \rangle$ . Evidently  $T_W \theta \geq \theta$  since for each  $C \in \ell_\infty^+(\alpha)$  we have  $T_W \theta(C) = W(c_1, 0) \geq 0$  as  $\theta(SC) = 0$ .

The defining characteristics of  $U^T$  are summarized based on the corresponding analysis in Marinacci and Montrucchio ([37]). The difference between our proofs is entirely concerned with the role played by Thompson criterion (T1) in our argument. Otherwise, our argument is quite similar to theirs.

## 4.2 The Order Interval $\langle \theta, U^T \rangle$

The definition of  $U^T$  is the first order of business in this subsection. We consider a Thompson aggregator,  $W$ . We have already specified the order interval’s bottom element is the zero-function,  $\theta$ . Note that it is trivially a  $\|\bullet\|_\alpha$  – continuous function and also belongs to  $C_\gamma^\alpha$  (by (T1)). Marinacci and Montrucchio [37] define the function  $U^T$  as follows:

$$U^T(C) = W(1, y_\alpha) \varphi_\gamma(C).$$

Here, the element  $y_\alpha > 0$  is the solution to  $W(1, y_\alpha) = \alpha^{-1/\gamma} y_\alpha$  (shown to exist in [37] using the additional properties (T5) – (T6)). It is straightforward to verify  $U^T \in (F_\gamma^\alpha)^+$ . Clearly  $U^T \geq \theta$  and  $U^T(C) > 0$  whenever  $C \neq 0$  and  $\|U^T\|_\gamma = W(1, y_\alpha) < +\infty$ . Furthermore,  $U^T \in (C_\gamma^\alpha)^+$  follows from its definition.<sup>20</sup>

The next result (again, see Marinacci and Montrucchio [37] for the proof) is critical to showing the Koopmans operator is a self-map on the order interval  $\langle \theta, U^T \rangle \subset (F_\gamma^\alpha)^+$ .

**Proposition 7** *If  $W$  is a Thompson aggregator satisfying (T5) and (T6), then  $T_W U^T \leq U^T$ .*

The construction of the order interval  $\langle \theta, U^T \rangle$  and verification that  $T_W \theta \geq \theta$  and  $T_W U^T \leq U^T$  means that the Koopmans equation has both a **subsolution**,  $\theta$ , and a **supersolution**,  $U^T$ . Zeidler ([46], p. 69) offers the following general existence principle in mathematics: *the existence of both a subsolution and a*

<sup>20</sup>Of course, this is true because the underlying  $\|\bullet\|_\alpha$  – norm is a (uniformly) continuous function from  $\ell_\infty^+(\alpha)$  into  $\mathbb{R}_+$ .

*supersolution yields the existence of a solution* to an operator equation. Zeidler’s point is that iteration of a monotone operator starting from either the subsolution as the initial seed or the supersolution as the initial seed converges to a solution for the operator equation. The solution found by iteration may differ according to the choice of the initial seed. The Tarski-Kantorovich Theorem illustrates Zeidler’s principle as the greatest element,  $\bar{x}$ , is a supersolution to the equation  $F(x) = x$  while the smallest element,  $\underline{x}$ , is a subsolution. We turn next to showing how this principle applies to the Koopmans operator and its corresponding functional equation.

### 4.3 Proof of the Marinacci-Montrucchio Recovery Theorem

The formal proof of the Marinacci-Montrucchio Recovery Theorem depends on verifying the Koopmans operator meets the conditions for invoking the Tarski-Kantorovich Fixed Point Theorem. There are three key requirements that must be checked in order to verify the Tarski-Kantorovich Theorem solves the Koopmans operator fixed point problem. First, the Koopmans operator for a given Thompson aggregator must be a self-map on the order interval  $\langle \theta, U^T \rangle$ . Second, this order interval must be a countably chain complete poset as a subset of the space  $(F_\alpha^\gamma)^+$ . Third, the Koopmans operator must be proven monotonically sup-inf preserving. *The latter property rests on the joint continuity assumption put in place by (T1) for Thompson aggregators.*<sup>21</sup> The monotonically sup-inf preservation property is the order continuity condition satisfied by the Koopmans operator. A successive approximation (indexed on the natural numbers) approach recovers at least one underlying utility function from the given Thompson aggregator.<sup>22</sup>

Marinacci and Montrucchio [37] prove their recovery theorem as an application of Tarski’s Fixed Point Theorem. Their result only relies on the fact that the Koopmans operator is a monotone self-map defined on a complete lattice given by the order interval  $\langle \theta, U^T \rangle \subset (F_\gamma^\alpha)^+$ . It does NOT require any order continuity property for their demonstration. However, their proof is, strictly speaking, nonconstructive. They claim to successively approximate the extremal fixed points whose existence is guaranteed by the Tarski Fixed Point Theorem. They obtain a *sequence* of approximate utility functions via iteration on the natural numbers from the initial seed,  $U^T$ . However, this does not follow directly from Tarski’s Theorem. Tarski’s original proof of his result does NOT use any form of successive approximation arguments.<sup>23</sup>

The Tarski-Kantorovich Theorem, by contrast, is constructive in the sense that successive approximation indexed on the natural numbers IS the underlying

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<sup>21</sup>This is **the** subtle difference between our version of the recovery theorem and Marinacci’s and Montrucchio’s theory.

<sup>22</sup>Proving there is a unique solution is deferred at this stage.

<sup>23</sup>Cousot and Cousot [14] provide a so-called constructive proof without monotonic sup-inf continuity. However, their argument employs transfinite induction. Echenique [17] simplifies their proof while maintaining a transfinite induction argument.

method. The monotonically sup-inf preserving (order continuity) property is the additional ingredient that allows us to know our iteration finds the largest fixed point by iterating (on the natural numbers) from the initial seed given by the top element of the order interval. Likewise, the smallest fixed point is found via iteration with initial seed given by the zero function, the bottom element of the order interval.

There is one other important formal difference between the hypotheses of the Tarski and Tarski-Kantorovich Theorems. The conditions on the domain  $X$  and on the self-map  $F$  for the Tarski-Kantorovich result are weaker than those in Tarski's Theorem. The underlying poset  $X$  is no longer assumed to be a complete lattice. Our underlying vector space  $F_\alpha^\gamma$  is not a complete lattice. However, it is a Banach lattice and an order interval such as  $\langle \theta, U^T \rangle$  turns out to be complete as a sublattice of  $F_\alpha^\gamma$  in its own right since the underlying vector space is also Dedekind complete. A monotonic sequence in this order interval is automatically order bounded and the sup or inf of such a sequence belongs to the order interval as well. This observation also lies underneath Marinacci and Montrucchio's Recovery Theorem via Tarski's original Fixed Point Theorem. The appearance of the order interval and restriction of the Koopmans operator to that set yields a monotone self-map on a complete lattice in their case. However, we only require our order interval to be a countably chain complete poset in our proof. This structure is critical to demonstrate extremal fixed points can be approximated by iteration indexed by the natural numbers. Successive approximations yields extremal fixed points. We easily verify the countably chain complete property since we actually have it in place since the order interval in our application continues to be the same complete lattice defined in Marinacci and Montrucchio's [37] arguments. But, the key technical point is that the underlying Riesz space,  $F_\alpha^\gamma$ , is, in fact, a  $\sigma$ -Dedekind complete space. That is, the sup and inf of each countable order bounded subset of this space belongs to the space.

The major advantage of our approach is the verification that successive approximation yields extremal solutions to the Koopmans equation. Marinacci and Montrucchio's approach requires a transfinite induction argument to describe those limiting functions as fixed points which are the largest or smallest fixed points in the order interval. They do not formally demonstrate that the Koopmans operator is order continuous although that is implied in their Recovery Theorem's hypotheses (as shown here). It is this latter property that is crucial for correctly inferring that iteration from the bottom and top elements of the order interval yield by successive approximations the smallest and largest fixed points in the underlying function space's positive cone,  $(F_\alpha^\gamma)^+$ . This, in turn, has ramifications for deducing semicontinuity properties of these special fixed points.

#### 4.4 The Recovery Theorem's Formal Proof

The application of the Tarski-Kantorovich Theorem to the Koopmans operator turns on verifying it is monotonically sup/inf-preserving on the order interval

$\langle \theta, U^T \rangle \subset (F_\alpha^\gamma)^+$  and that order interval is also a countably chain complete set.

**Proposition 8** *Suppose  $W$  is a Thompson aggregator satisfying (T5) and (T6). Then the associated Koopmans operator is a monotonically sup/inf preserving self-map on  $\langle \theta, U^T \rangle$ .*

**Proof.** Lemma 5 implies  $T_W$  is a monotone operator. It is obvious that  $T_W \theta \geq \theta$  and  $T_W U^T \leq U^T$  follows from Proposition 7.

Suppose  $\{U^N\} \equiv \{U^N\}_{N=1}^\infty$  is a sequence of  $\|\bullet\| - \gamma$  norm bounded functions in the order interval  $\langle \theta, U^T \rangle \subset (F_\alpha^\gamma)^+$ . Clearly both the sup and inf of this sequence exist as elements of  $\langle \theta, U^T \rangle$ . This implies  $\{U^N\}$  is a countably chain complete set in  $\langle \theta, U^T \rangle$  provided it is a chain. Therefore,  $\langle \theta, U^T \rangle$  is a countably chain complete poset follows immediately as  $\{U^N\}$  may be an arbitrarily chosen countable chain in  $\langle \theta, U^T \rangle$ .

The order interval  $\langle \theta, U^T \rangle$  evidently contains a smallest and largest element. Now suppose  $\{U^N\}$  is any monotone increasing sequence of functions in  $\langle \theta, U^T \rangle$ . By countable chain completeness, we find  $\bigvee U^N$  exists since each  $U^N \leq U^T$ . Hence, there is a function  $U = \bigvee U^N \in \langle \theta, U^T \rangle$ . In fact,  $U^N \nearrow U$  pointwise on  $\ell_\infty^+(\alpha)$ . That is  $\lim_{N \rightarrow \infty} U^N(C) = U(C)$  for each  $C \in \ell_\infty^+(\alpha)$ . Since  $W$  is increasing in its second argument and *continuous in its second argument*, (T1) implies for each  $C \in \ell_\infty^+(\alpha)$  the following equalities:

$$\begin{aligned} \bigvee [T_W U^N](C) &= \bigvee W(c_1, U^N(SC)) \quad (\text{by definition of } T_W) \\ &= \lim_N W(c_1, U^N(SC)) \quad (\text{by the monotone property of } W \text{ in (T1)}) \\ &= W\left(c_1, \lim_N U^N(SC)\right) \quad (\text{by continuity of } W \text{ in (T1)}) \\ &= W(c_1, U(SC)) \\ &= T_W\left(\bigvee U^N\right)(C). \end{aligned}$$

Hence, the Koopmans operator is monotonically sup-preserving. Apply the analogous argument for monotone decreasing sequences  $\{U^N\}$ , bounded below by the zero function. This shows that  $T_W$  is also monotonically inf-preserving. Hence, the Koopmans operator is monotonically sup/inf-preserving. ■

This Proposition's proof seemingly depends only on assumption (T1). However, the other properties come into play when verifying  $T_W$  is a monotone self map on the order interval  $\langle \theta, U^T \rangle \subset (F_\alpha^\gamma)^+$ . *Our take on the Marinacci-Montrucchio Recovery Theorem proof appears below.*

**Proof.** Iterate  $T_W$  using  $U^T$  as the initial seed. That is, for each natural number,  $N$ , let

$$U^N = T_W U^{N-1} \text{ and } U^0 \equiv U^T.$$

Clearly for each  $N \geq 1$ ,

$$\theta \leq U^N \leq U^{N-1} \leq \dots \leq U^1 \leq U^T.$$

Hence, there is a function  $U^\infty$  such that

$$U^\infty = \bigwedge_N U^N \in \langle \theta, U^T \rangle$$

since  $\langle \theta, U^T \rangle$  is countably chain as a subset of  $(F_\alpha^\gamma)^+$ .

The function  $U^T$  is  $\|\bullet\|_\alpha$ -continuous on  $\ell_\infty^+(\alpha)$ . Hence, since, by (T1),  $W$  is a continuous function on  $\mathbb{R}_+^2$ , the function  $U^1 = T_W U^T$  is also a  $\|\bullet\|_\alpha$ -continuous function on  $\ell_\infty^+(\alpha)$ , and so on for each  $U^N$ . Hence,  $U^\infty$  is a  $\|\bullet\|_\alpha$ -upper-semicontinuous real-valued function on  $\ell_\infty^+(\alpha)$  as it is the pointwise infimum of continuous functions. Proposition 8 shows that  $T_W$  is monotonically sup-inf-preserving. Therefore,  $T_W$  satisfies the hypotheses of the Tarski-Kantorovich Theorem. Hence, we may conclude by that Theorem that  $U^\infty$  is a fixed point of the Koopmans operator. That is,

$$T_W U^\infty = U^\infty.$$

■

The Tarski-Kantorovich Theorem actually yields a bit more information.

**Corollary 9** *The fixed point  $U^\infty = \bigwedge_N T_W^N U^T$  is the largest fixed point of the Koopmans operator in  $\langle \theta, U^T \rangle$ .  $U^\infty$  is a  $\|\bullet\|_\alpha$ -upper-semicontinuous real-valued function on  $\ell_\infty^+(\alpha)$ . There is also a smallest fixed point in  $\langle \theta, U^T \rangle$ , denoted  $U_\infty = \bigvee_N T_W^N \theta$  found by iterating  $T_W$  with the initial seed,  $\theta$ , the zero function. Moreover,  $U_\infty$  is a  $\|\bullet\|_\alpha$ -lower-semicontinuous real-valued function on  $\ell_\infty^+(\alpha)$ .*

Let  $\text{fix}(T_W)$  denote the nonempty set of fixed points belonging to our Koopmans operator in the order interval  $\langle \theta, U^T \rangle$ . Quoting Balbus, et al (see [6], Theorem 7, p. 109) we arrive at another result.

**Corollary 10**  *$\text{fix}(T_W) = \langle U_\infty, U^\infty \rangle$  is a countably chain complete poset.*

It follows that **IF**  $U^\infty = U_\infty \equiv U^*$ , then  $U^* \in (F_\alpha^\gamma)^+$  is the unique  $\|\bullet\|_\alpha$ -continuous  $\varphi_\gamma$ -bounded real-valued function in the order interval  $\langle \theta, U^T \rangle$  satisfying the Koopmans equation when  $W$  is a Thompson aggregator. That is, in this situation  $U^* \in (C_\alpha^\gamma)^+$  as well! *Uniqueness of the solution in the larger space  $(F_\alpha^\gamma)^+$  implies that the solution is also a  $\|\bullet\|_\alpha$ -continuous and  $\varphi_\gamma$ -bounded real-valued function!* The interesting problem at this point is to provide conditions under which there is a unique  $\|\bullet\|_\alpha$ -continuous and  $\varphi_\gamma$ -bounded solution to this aggregator's Koopmans equation.

This uniqueness question is subtle. There are two issues. First, the extremal fixed points are never identical on the entire domain,  $\ell_\infty^+(\alpha)$  whenever  $W(0, 0) = 0$ . This property holds for the CES and KDW aggregators. The following example illustrates this point using the CES aggregator. Set  $\alpha = 1$ . Note that

there is a unique  $y^* > 0$  such that  $W(1, y^*) = y^*$ . Choose a natural number  $N$ . Compute  $T_W^N \theta(C)$  and evaluate this expression at  $C = 0$  to obtain:

$$T_W^N \theta(0) = W(0, 0) = 0.$$

Hence, passing to the limit we find  $U_\infty(0) = 0$ . On the other hand, computation of  $U^\infty(0)$ , proceeds as follows since can compute the iterates directly when  $W$  is a CES aggregator:

$$\begin{aligned} T_W^0 U^T(0) &\equiv U^T(0) = y^* \varphi_\gamma(0) = y^* \text{ as } \varphi_\gamma(0) = 1; \\ T_W^1 U^T(0) &= W(0, y^*) = \beta (y^*)^\rho; \\ T_W^2 U^T(0) &= W(0, W(0, y^*)) = \beta [\beta (y^*)^\rho]^\rho = \beta^{1+\rho} (y^*)^{\rho^2}; \\ &\vdots \\ T_W^N U^T(0) &= W(0, W(0, W(0, \dots, W(0, y^*))) = \beta^{(1+\rho+\rho^2+\dots+\rho^{N-1})} (y^*)^{\rho^N}. \end{aligned}$$

Clearly  $\rho^N \rightarrow 0$  as  $N \rightarrow \infty$  implies  $\lim_N (y^*)^{\rho^N} = 1$  and

$$U^\infty(0) = \lim_{N \rightarrow \infty} T_W^N U^T(0) = \beta^{\left(\frac{1}{1-\rho}\right)} > 0,$$

and hence,  $U^\infty(0) > U_\infty(0) = 0$ . The extremal fixed points of the Koopmans operator cannot agree on the entire domain,  $\ell_\infty^+$ . The Koopmans operator, defined for all consumption sequences in  $\ell_\infty^+$ , is **NOT** uniquely determined by the aggregator function! However, this does not mean we cannot say something useful about the subset of consumption sequences where the extremal fixed points deliver the same utility value. We answer this problem in the next section.

The second issue concerns the interpretation of multiple solutions to the operator equation when the set of fixed points is itself a nontrivial order interval. The Recovery Theorem suggests it is possible for the Koopmans equation to yield multiple solutions for a given Thompson aggregator. Are all of those solutions in  $\text{fix}(T_W) = \langle U_\infty, U^\infty \rangle$  ordinally equivalent? If so, the lack of uniqueness is not an economic issue — each of the solutions may be taken as representative of the underlying intertemporal preference relation. However, if the elements in  $\text{fix}(T_W)$  are NOT ordinally equivalent, then some (most) are spurious solutions that do NOT represent the underlying preference relation. It suffices to exam the relationship between the top and bottom elements of  $\text{fix}(T_W)$ . The resolution of this question is that  $U_\infty = U^\infty$  if and only if these two the elements of  $\text{fix}(T_W)$  are ordinally equivalent. In that situation there is a unique solution in  $\langle \theta, U^T \rangle$ .

Ordinal equivalence of  $U^\infty$  and  $U_\infty$  simply means that there is a monotonically increasing transformation  $\Phi$  such that  $U^\infty = \Phi(U_\infty)$ . Of course the domain and range of  $\Phi$  must be appropriate. Put differently, for each  $C \in \ell_\infty^+(\alpha)$ ,  $U^\infty(C) = \Phi(U_\infty(C))$ . If  $U^\infty$  and  $U_\infty$  are ordinally equivalent utility functions, then their aggregators are also Koopmans equivalent (see the Appendix). That is, there are (in this case) **Thompson** aggregators  $W^*$  belonging to  $U^\infty$  and



$W$  belonging to  $U_\infty$  such that for each  $(x, y) \in \mathbb{R}_+^2$  :

$$W^*(x, y) = \Phi [W(x, \Phi^{-1}(y))] .$$

The assumption that  $T_W U_\infty = U_\infty$  and  $T_W U^\infty = U^\infty$  implies  $W^* = W$  must hold. That is, these two utility functions are derived from the same aggregator function so  $W^* = W$ . Hence,

$$W(x, y) = \Phi [W(x, \Phi^{-1}(y))] .$$

obtains if and only if  $\Phi$  is the identity map. Hence,  $U^\infty = U_\infty$  follows and the Koopmans equation has a unique solution in  $\langle \theta, U^T \rangle$ . This is reasonable — there are spurious solutions whenever  $\text{fix}(T_W)$  has the property that the top element is not a monotone increasing transformation of the bottom element. In this case ordinal equivalence fails and one of those two elements must be a spurious solution. Indeed, there may be other spurious solutions in the order interval  $\langle U_\infty, U^\infty \rangle$ . Hence, showing that the operator equation has a unique solution in  $\langle \theta, U^T \rangle$  tells us there are no spurious solutions to the Koopmans equation (at least in  $\langle \theta, U^T \rangle$ ) and we may conclude that THE fixed point is THE TRUE solution providing a recursive utility representation for the underlying preference relation.

## 5 Uniqueness Theory: The $u_0$ – Concave Operator Method

The Marinacci and Montrucchio Recovery Theorem’s proof exploited the intrinsic monotonicity property of the Koopmans operator. This feature, together with a stronger condition,  $u_0$  – *concavity*, turns out to imply the Koopmans equation has a unique solution in  $(C_\alpha^+)^+$ . This method for proving uniqueness emphasizes order properties embedded in Thompson aggregators and their Koopmans operators when the aggregators are assumed to be **jointly concave** in both arguments. It is important to recall this joint concavity property is a common characteristic of CES and KDW Thompson aggregators. We show how to exploit this fundamental concavity property from these examples to obtain a uniqueness theorem without appealing to any form of contraction theorem. We stress that Marinacci and Montroccio followed the generalized contraction methodology in producing their uniqueness theory for the Thompson case. Our purpose here is to take a different route sympathetic to our stress on order properties and showcase the advantages of  $u_0$  – *concave operators* in the present context.

A technical problem confronts us in our search for a unique solution to the Koopmans equation belonging to a Thompson aggregator. Consumption sequences where there is no consumption for at least one time period appear to inhibit the verification of conditions sufficient for a unique solution. This is clear from the CES example in the last section where  $U_\infty(0) < U^\infty(0)$ . At a deeper level, we seek the uniqueness of a non-trivial solution. Here, by

nontrivial, we mean a non-zero solution to the Koopmans equation. We are insured there exist at least one nontrivial solution by the existence theorem. The problem with consumption sequences which are zero at some time index arises whether we adopt the Thompson metric approach championed by Marinacci and Montrucchio, or the  $u_0$ -concave operator approach presented in this paper.<sup>24</sup> More about this issue shortly.

## 5.1 $u_0$ -Concave Operators and Uniqueness Theory for Nonlinear Operators

We begin with an abstract treatment motivated by the fact that our Koopmans operator is a self-map on the positive cone,  $(F_\alpha^\gamma)^+$ , which is a subset of a real Banach space. We review the basic uniqueness theory for an abstract nonlinear operator defined on the positive cone  $P$  of a real Banach space. Our purpose is to apply results obtained for uniqueness of solutions without relying on a (generalized) contraction mapping theorem, but instead drawing on features derived from the presence of a partial order on elements of the underlying real Banach space. Our exposition draws on Krasnosel'skiĭ and Zabreĭko [29] as well as Krasnosel'skiĭ [27]. The latter reference is the seminal work on concave operators and also introduced the stronger  $u_0$ -concavity property used here. We also present new results obtained by Liang, Wang, and Li (hereafter, Liang et al) [31]. Closely related references include Guo and Lakshmikantham [19] and Guo, Cho, and Zhu [20]. Coleman [13] introduced concave operator techniques into the economics literature.

### 5.1.1 The Krasnosel'skiĭ and Zabreĭko Theorem

Fix a *non-zero* element, denoted by  $u_0$ , in a cone  $P$ .<sup>25</sup> Let  $\theta$  denote the zero element in  $P$ . An operator  $A : P \rightarrow P$  is called  $u_0$ -concave on  $P$  if for each non-zero element  $x \in P$  there are positive scalars  $a(x)$  and  $b(x)$  such that

$$a(x)u_0 \leq Ax \leq b(x)u_0, \quad (8)$$

and if for each  $x \in P$  with  $a_1(x)u_0 \leq x \leq b_1(x)u_0$ , for some  $a_1, b_1 > 0$ , we have

$$A(tx) \geq [1 + \eta(t, x)]tAx \text{ for } 0 < t < 1, \quad (9)$$

where  $\eta(t, x) > 0$ .

The restrictions  $a_1(x)u_0 \leq x \leq b_1(x)u_0$  and  $u_0 \neq \theta$  taken together mean that  $x > \theta$ , but not necessarily that  $x \gg \theta$  (i.e., that  $x \in \text{int}(P)$ ). Inequality(9) implies  $A(tx) > tAx$  holds. This falls short of asserting  $A(tx) \gg tAx$  — that is, the vector difference  $A(tx) - tAx \in \text{int}(P)$ . In general this property cannot

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<sup>24</sup>Marinacci and Montrucchio [37] adjust the domain of their utility functions to rule out troublesome consumption sequences where there may be periods with zero consumption. We do this too, but in a different maner.

<sup>25</sup>Recall, this cone must be nonempty, convex, and norm-closed.

even be considered unless the underlying cone is solid.<sup>26</sup> Hence, we work with the inequalities expressed in (9). This turns out to be sufficient for settling our uniqueness question, at least in the most abstract framework.

The main Krasnosel'skiĭ and Zabreĭko result ([29], Theorem 46.1, p. 290) is stated below along with its proof to keep our presentation reasonably self-contained.<sup>27</sup>

**Theorem 11** (*Krasnosel'skiĭ and Zabreĭko*). *Let  $A$  be a monotone operator which is  $u_0$ -concave on  $P$ . Then the equation*

$$Ax = x$$

*has at most one non-zero solution in  $P$ .*

**Proof.** Suppose there are two nontrivial solutions in  $P$ :  $Ax_1 = x_1$  and  $Ax_2 = x_2$ . There are two cases:  $x_1 \geq x_2$  and  $x_1 \not\geq x_2$ . Consider the second case first.

Since  $A$  is  $u_0$ -concave, there are positive numbers  $a(x)$  and  $b(x)$  such that

$$\begin{aligned} a(x_1)u_0 &\leq Ax_1 \leq b(x_1)u_0; \\ a(x_2)u_0 &\leq Ax_2 \leq b(x_2)u_0. \end{aligned}$$

Therefore,

$$x_1 = Ax_1 \geq a(x_1)u_0 \text{ and } x_2 = Ax_2 \leq b(x_2)u_0$$

implies  $x_1 \geq tx_2$  for some small, but positive, scalar  $t$ . Hence, there is a  $t^* \in (0, 1)$  such that  $x_1 \geq t^*x_2$  and  $x_1 \not\geq tx_2$  for  $t > t^*$ .

Condition (9) implies

$$A(t^*x_2) \geq (1 + \eta)t^*Ax_2$$

where  $\eta > 0$ . Since  $A$  is monotone, we obtain the following inequalities:

$$x_1 = Ax_1 \geq A(t^*x_2) \geq (1 + \eta)t^*Ax_2 = (1 + \eta)t^*x_2.$$

This implies  $x_1 \geq (1 + \eta)t^*x_2 > t^*x_2$ , which contradicts the defining condition determining  $t^*$ . A similar argument applies to the case  $x_1 \geq x_2$ . Therefore, there is at most one nontrivial solution of  $Ax = x$  in the cone  $P$ . ■

It is of some interest to note that this result does not require ANY additional assumptions on the cone  $P$ ! For example,  $P$  is not assumed to be a normal cone. A cone  $P$  is **normal** provided  $\theta \leq x \leq y$  implies  $\|\theta\| \leq \mathcal{N}\|y\|$  and the constant  $\mathcal{N}$  does not depend on the choice of  $x$  or  $y$ . The result emphasizes order theoretic ideas based on the partial ordering of the underlying function space as defined by the elements of the cone  $P$ . Topological considerations enter through the underlying Banach space structure (which explicitly determines the

<sup>26</sup>It happens that our cones are solid. But, the general mathematical theory does not impose that particular requirement.

<sup>27</sup>Krasnosel'skiĭ ([27], p. 188) presents the first formulation of this result within the context of nonlinear eigenvalue problems.

norm-topology) whereby  $P$  is a closed set. The positive cone is normal in our setup.

Krasnosel'skiĭ and Zabreiko's Theorem imposes two conditions on the operator  $A$  — it is monotone and  $u_0$ -concave. The Koopmans operator is monotone, so the only task in applying their theorem in our setting concerns verification that it is a  $u_0$ -concave operator.

*There are two critical estimates that must be satisfied to verify  $A$  is a  $u_0$ -concave operator in the general abstract setting. The method of Liang et al [31] offers a set of sufficient conditions that, once (8) is verified, implies (9) is also satisfied. A second type of concave operator can prove useful in verifying the  $u_0$ -concave property. Liang et al's ([31]) contribution turns on using this second idea about expressing concavity for abstract operators.*

### 5.1.2 The Liang, Wang and Li Ordered Concave Operator Theorem

Fix the function  $u_0 > \theta$  with  $u_0 \in P$  as above. Define

$$P_{u_0} = \{x \in P : \exists a(x), b(x) > 0 \text{ such that } a(x)u_0 \leq x \leq b(x)u_0\}. \quad (10)$$

The set  $P' \subseteq P$  serves as the range of the operator  $A$  acting on the cone  $P$ . This range set  $P'$  may also be a cone, but this is not a formal requirement. What matters is that  $P'$  inherits the partial order induced by the cone,  $P$ . Assume further that  $P'$  is a convex subset of  $P$ .

Condition (8) differs from (10). The former uses the image of  $x$  under the mapping  $A$ , whereas the latter employs the domain point,  $x$ . The set  $P_{u_0}$  is the operator's "target." This is why we must establish (8) in order to verify  $A$  has the  $u_0$ -concavity property.

**Definition 12**  $A : P \rightarrow P'$  is an **ordered concave operator** if for each  $x, y \in P$  with  $x \geq y$  and  $t \in [0, 1]$ :

$$A(tx + (1-t)y) \geq tAx + (1-t)Ay. \quad (11)$$

The convexity of the range space is critical for the inequality above to make sense.

An ordered concave operator need NOT automatically be a  $u_0$ -concave operator. Some authors (e.g. [19] and [20]) identify the terms concave and ordered concave operators. We prefer to keep the prefix "ordered" in place as a reminder the condition  $x \geq y$  has an important role to play in this theory.

The following lemma is crucial. It is implicit in Liang et al's proof ([31], Lemma 4, p. 579). We note that  $P_{u_0} \subset P$ .

**Lemma 13** Suppose  $A : P \rightarrow P_{u_0}$  is an ordered concave operator. Then for each  $x \in P$  there is a positive number  $\mu(x)$  such that

$$A\theta \geq \mu(x)Ax. \quad (12)$$

**Proof.** Let  $x \in P$ . Then  $Ax \in P_{u_0}$  by assumption. By the definition of  $P_{u_0}$  there are numbers  $a(Ax) > 0$ ,  $b(Ax) > 0$  such that

$$a(Ax)u_0 \leq Ax \leq b(Ax)u_0.$$

In particular,  $x = \theta \in P$ . Thus, the above inequality for  $\theta$  is also true and reads

$$a(A\theta)u_0 \leq A\theta \leq b(A\theta)u_0.$$

Combining both sets of inequalities as follows we have

$$A\theta \geq a(A\theta)u_0 = \left(\frac{a(A\theta)}{b(Ax)}\right)b(Ax)u_0 \geq \mu(x)Ax$$

where

$$\mu(x) \equiv \left(\frac{a(A\theta)}{b(Ax)}\right) > 0.$$

■

We notice that the function  $\mu(x)$  depends on the fixed operator,  $A$ , and the zero function,  $\theta$  as  $x$  varies in  $P$ .

**Theorem 14** (Liang et al [31]) *Suppose  $A : P \rightarrow P_{u_0}$  is an ordered concave operator. Then  $A$  is a  $u_0$ -concave operator.*

**Proof.** For each  $0 < t < 1$ : the ordered concavity condition implies, since  $x \geq \theta$ , that

$$\begin{aligned} A(tx) &= A(tx + (1-t)\theta) \\ &\geq tAx + (1-t)A\theta \\ &\geq tAx + (1-t)\mu(x)Ax \text{ (by (12))} \\ &= t\left(1 + \frac{(1-t)\mu(x)}{t}\right)Ax. \end{aligned}$$

Now set

$$\eta(t, x) = \frac{(1-t)\mu(x)}{t} > 0$$

since  $\mu(x) > 0$ . Hence,  $A$  is a  $u_0$ -concave operator as we have shown

$$A(tx) \geq (1 + \eta(t, x))tAx.$$

■

The practical impact of this result is plain — check that an operator is a  $u_0$ -concave operator on the cone  $P$  by verifying the sufficient condition in Liang, Wang, and Li's Theorem! This effectively means checking that the given operator satisfies condition (12). The application of  $u_0$ -concave operator theory to the Koopmans operator,  $T_W$ , turns on checking that condition. Doing so means there must be a clear choice of the cone,  $P$ , selection of the point  $u_0$ , verification that  $T_W U$  belongs to the corresponding set  $P_{u_0}$  whenever  $U \in P$ , and checking  $T_W$  is an ordered concave operator. Additional restrictions on the underlying commodity space must be imposed to show the Koopmans operator is a  $u_0$ -concave operator. We offer an economically motivated condition towards this end next.

## 5.2 The Commodity Space: Minimal and Maximal Consumption Sequences

We restrict the structure of the commodity space to  $\ell_\infty^+$  for the uniqueness theory. There is a technical reason for doing so — the order intervals capturing minimal and maximal consumption constraints must be *shift invariant* in the recursive utility framework. Marinacci and Montrucchio [37] use a different shift invariant domain that, unlike ours, accomodates growing economies. Our choice of domain is motivated by its intuitive economic basis as well as for its technical convenience in demonstrating the Koopmans operator satisfies the conditions for Liang et al's result [31].

A constant vector in  $\ell_\infty$  is denoted by  $\mathbf{x}_{con} = (x, x, x, \dots)$ , where  $x$  is a real number. Let  $\varepsilon_{con} = (\varepsilon, \varepsilon, \varepsilon, \dots) \in \ell_\infty^+$  when  $\varepsilon > 0$ . Here,  $\varepsilon$  is interpreted as the **minimal consumption** requirement at any time,  $t$ , where  $t = 1, 2, \dots, t$ . This is an economically motivated restriction used to define the commodity space

$$\ell_\infty^+(\varepsilon) = \{C \in \ell_\infty^+ : C \geq \varepsilon_{con}\} \subset \ell_\infty^{++},$$

where  $\ell_\infty^{++}$  is the **norm interior** of  $\ell_\infty^+$ . This interior is nonempty as the positive cone contains an order unit,  $e \equiv \mathbf{e}_{con} = (1, 1, 1, \dots)$ . We also note  $\{C \in \ell_\infty^+ : \inf_t c_t > 0\} \subset \ell_\infty^{++}$ . We employ the obvious notation  $C = \{c_t\}$  with  $t = 1, 2, \dots$

We note that the space  $\ell_\infty^+(\varepsilon)$  is a topological subspace of the Banach space  $(\ell_\infty^+, \|\bullet\|_\infty)$ . Clearly the sequence  $\varepsilon_{con} = \varepsilon e$ .

A **maximum consumption** constraint is also placed on consumption sequences for the purpose of proving a uniqueness theorem. This shows up when we confine our analysis to order intervals in  $\ell_\infty^+$  of the form  $[ae, be]$  where  $0 < a < b < \infty$ . The subset of  $\ell_\infty^+$  consisting of all sequences in this order interval is denoted by  $K_{ab} \subset \ell_\infty^+$ . In fact,  $K_{ab} \subset \ell_\infty^{++}$ . Thus,  $C \in K_{ab}$  implies  $0 < a \leq c_t \leq b$  for each  $t$ . A maximum consumption constraint is reasonable in some growth models, such as the standard one-sector Ramsey model with a positive maximum sustainable capital stock — consumption can NEVER exceed the output produced using that capital stock. In our uniqueness theory we treat the numbers  $a$  and  $b$  as parameters that are varied (subject to  $0 < a < b < \infty$ ). The order intervals  $K_{ab}$  incorporate both minimal consumption and a maximal consumption sequence. These sequences play a fundamental role in demonstrating how to adapt  $u_0$  – *concave* operator to our problem. The natural bounds in consumption sequences in  $K_{ab}$  are critical in constructing bounds on utility values that are independent of any particular  $C \in K_{ab}$ .<sup>28</sup>

It is reasonable to conjecture  $\cup_{a,b} K_{ab} = \ell_\infty^{++}$ . However, this is not true. A bounded consumption sequence of positive terms that converges to 0 will not be in  $K_{ab}$  for any  $a > 0$ . Define the set  $\ell_\infty^> \subset \ell_\infty^+$ , by

$$\ell_\infty^> = \left\{ C \in \ell_\infty^+ : \liminf_{t \rightarrow \infty} c_t > 0 \right\}.$$

<sup>28</sup>The countable increasing family of subsets  $K_n = [\frac{1}{n}e, ne]$  for each natural number,  $n$ , form a subcollection of our  $\{K_{ab}\}$ . There appear to be some technical advantages to using the larger collection in our proofs than the countable collection indexed on the natural numbers.

Let  $\mathcal{K} = \{K_{ab}\}$  for  $0 < a < b < \infty$  be the family of all order intervals of the form  $K_{ab} \subset \ell_\infty^{++}$ . This family covers  $\ell_\infty^>$ .

**Lemma 15**  $\cup_{a,b} K_{ab} = \ell_\infty^>$ .

**Proof.** Suppose  $C \in K_{ab}$  for some  $0 < a < b$ , then  $0 < a \leq c_t \leq b$  for each  $t$  and thus  $C \in \ell_\infty^{++}$  and  $C \in \ell_\infty^>$  as well (note that the limit inferior of this sequence always exists and is obviously positive in this case).

Now suppose that  $C \in \ell_\infty^>$ . Then by assumption,  $c_t \leq b$  for some  $b > 0$  since  $C$  is bounded. Moreover, let  $0 < a' = \liminf_{t \rightarrow \infty} c_t$ . By definition of the limit inferior,  $c_t \geq a'$  for all  $t$  except at most for finitely many indexes, denoted  $t_1, t_2, \dots, t_n$ . That is,  $c_{t_k} < a'$  for  $k = 1, 2, \dots, n$ . Set  $a = \min\{c_{t_1}, c_{t_2}, \dots, c_{t_n}\}$  and take  $a = a'$  if this finite family of consumption terms is empty. Note that  $a > 0$  since  $c_t > 0$  for each  $t$  and the minimum is taken with respect to a finite family. Therefore  $C \in K_{ab}$ . ■

Each order interval  $K_{ab}$  is shift invariant. Recall the shift operator  $S : \ell_\infty \rightarrow \ell_\infty$  according to the formula:

$$SC = (c_2, c_3, \dots).$$

The shift operator has the property  $S(K_{ab}) \subseteq K_{ab}$ . That is,  $K_{ab}$  is a shift invariant subset of this commodity space. The lemma implies  $\ell_\infty^>$  is shift invariant.

Our commodity space restriction is similar to the ones imposed by Marinacci and Montrucchio [37] for their uniqueness theory. Specifically, they employ the following sets in their uniqueness theory:

$$\begin{aligned} C^{\gamma,1} &= \{C \in \ell_\infty^+ : W(c_1, 0) > 0\}; \\ A^{\gamma,1} &= \left\{C \in \ell_\infty^+ : \limsup_{t \rightarrow \infty} c_t < +\infty \ \& \ \liminf_{t \rightarrow \infty} W(c_t, 0) > 0\right\}. \end{aligned}$$

The possibility that  $\liminf_t c_t = 0$  is excluded in the last set: continuity of the aggregator would imply  $\lim_t W(c_t, 0) = 0$  as  $W(0, 0) = 0$  can occur in cases such as the CES and KDW Thompson aggregators.<sup>29</sup> However, if  $\liminf_t c_t > 0$ , then in those examples,  $\liminf_t W(c_t, 0) > 0$  holds as well. This implies for the CES and KDW aggregators that:

$$\left\{C \in \ell_\infty^{++} : 0 < \liminf_{t \rightarrow \infty} c_t \leq \limsup_{t \rightarrow \infty} c_t < +\infty\right\} \subset A^{\gamma,1}.$$

It is easy to show (adapting the lemma's proof) that

$$\left\{C \in \ell_\infty^{++} : 0 < \liminf_{t \rightarrow \infty} c_t \leq \limsup_{t \rightarrow \infty} c_t < +\infty\right\} = \ell_\infty^> = \cup K_{a,b}.$$

Clearly equality holds between these two sets since consumption sequences are always bounded given  $K_{ab}$ . Hence, our commodity space restrictions may exclude *some* cases available in their theory, but we argue our approach is based on

<sup>29</sup>The Thompson properties (T1) – (T4) do not formally require  $W(0, 0) = 0$ .

the more economically appealing abstraction motivated by minimal consumption at each time in the sets  $K_{ab}$ .

The class of utility functions for which at most one solution to the operator equation,  $T_W U = U$ , defined by the Koopmans operator, must be specified clearly. We take the set  $\ell_\infty^+$  as the domain for our possible utility function solution(s). Let

$$B = \{U : \ell_\infty^+ \rightarrow \mathbb{R}_+ \text{ such that } U \text{ is } \varphi_\gamma \text{ - bounded}\}.$$

The usual pointwise partial order is imposed and denoted by  $\geq$ . The set  $B$  is identical to the set  $F_\gamma^\alpha$  when  $\alpha = 1$ . We use this alternative notation here since it is less cumbersome and reminds us that geometric consumption growth rates are excluded from the economy under consideration. The underlying economy is subject to diminishing marginal returns and there is a maximum sustainable capital stock that ultimately bounds consumption sequences.

The corresponding positive cone of this Banach space is

$$B^+ = \{U : \ell_\infty^+ \rightarrow \mathbb{R}_+ \text{ such that } U \text{ is } \varphi_\gamma \text{ - bounded and } U \geq \theta\}.$$

Cone  $B^+$  is a normal cone since it contains an order unit, namely the function  $U(C) \equiv 1$  for each  $C \in \ell_\infty^+$ .

We also consider the set:

$$B_{T_W \theta}^+ = \{U : \ell_\infty^+ \rightarrow \mathbb{R}_+ \text{ such that } U \text{ is } \varphi_\gamma \text{ - bounded and } U \geq T_W \theta\}.$$

While,  $\varphi_\gamma$  can be any weight function for now — it will be fixed as  $\varphi_\gamma(c) = (1 + \|C\|_\infty)^{1/\gamma}$  as in Marinacci and Montrucchio's construction. We set  $\alpha = 1$  and let  $\|C\|_\infty$  denote the standard sup norm on  $\ell_\infty$ . We know from our Theorem 6 and Marinacci and Montrucchio's Recovery Theorem that there is a norm upper-semicontinuous real-valued function  $U^\infty \in B^+$  such that  $T_W U^\infty = U^\infty$ . *Our goal is to show that  $U^\infty$  is the unique solution to the Koopmans equation, at least among nonnegative  $\varphi_\gamma$ -bounded functions defined on  $\ell_\infty^+$ .*

Our methods depend on the shift invariance of the order intervals in the collection  $\{K_{ab}\}$  since we consider  $U^\infty|_{K_{ab}}$ , the restriction of  $U^\infty$  to the domain  $K_{ab}$ . We ask that  $U^\infty|_{K_{ab}}$  remain a solution on each such order interval. Hence, for a point  $C \in K_{ab}$  we require

$$U^\infty(C) = W(c_1, U^\infty(SC))$$

to hold. This equation only makes sense if  $SC \in K_{ab}$  as well. This shift invariance property does NOT extend to order intervals built for growing economies. Let  $\omega = (\alpha, \alpha^2, \alpha^3, \dots)$  where the growth factor  $\alpha > 1$ . Consider the order intervals of the form  $[\frac{1}{n}e, n\omega] \subset \ell_\infty^+(\alpha)$ . The shift operator is not invariant on such an order interval. Set  $n = 1$  and consider  $S\omega = (\alpha^2, \alpha^3, \alpha^4, \dots)$ . Then  $\alpha^2 > \alpha > 1$  implies  $S\omega \notin [e, \omega]$ .

Our methods cannot be improved to include consumption sequences that vanish at one, or more time periods. Consumption sequences with one, or



more, zero consumption periods are mathematically inconvenient — they are not economically important in widely used models (unless survival is a critical issue over the planning horizon). The advantage of working with some form of minimal consumption requirements is the following: for each  $\varepsilon > 0$ ,

$$\inf_{C \in \ell_{\infty}^+(\varepsilon)} T_W \theta(C) = W(\varepsilon, 0) \equiv W_{\varepsilon} > 0$$

since  $W$  is a Thompson aggregator (property (T4)) and there is a minimal consumption at each time given by  $\varepsilon > 0$ . In particular we note that for sequences restricted to  $K_{ab}$  there are upper and lower bounds of the following type (by  $W$  monotone and Thompson):

$$\begin{aligned} \inf_{C \in K_{ab}} T_W \theta(C) &= W(a, 0) > 0; \\ \sup_{C \in K_{ab}} T_W \theta(C) &= W(b, 0) < \infty. \end{aligned}$$

Let

$$B^+|K_{ab} = \{U \in B^+ : U|K_{ab} \rightarrow \mathbb{R}_+ \text{ and } U \geq \theta\}.$$

This is just the set of trial utility functions in  $B^+$  restricted, or cut down, to the domain  $K_{ab}$ . The notation  $U|K_{ab}$  is shorthand for  $U : K_{ab} \rightarrow \mathbb{R}_+$  and  $U$  is  $\varphi_{\gamma}$ -bounded.

## 6 $T_W$ is $T_W \theta$ – Concave

We prove that the Koopmans equation has at most one solution in the set of  $\varphi_{\gamma}$ -bounded functions defined on the set  $\ell_{\infty}^>$ . Our methods do NOT allow us to prove there is at most one solution for  $\varphi_{\gamma}$ -bounded functions defined on the positive cone  $\ell_{\infty}^+$  or its norm interior,  $\ell_{\infty}^{++}$ . Our solution to this uniqueness problem builds on the existence of a particular extremal solution,  $U^{\infty}$ , which is a  $\varphi_{\gamma}$ -bounded function on  $\ell_{\infty}^+$ . The solution  $U^{\infty}$  remains a solution to the Koopmans equation when its domain of definition is restricted to a shift invariant subset of the positive cone's sup norm interior,  $\ell_{\infty}^{++}$ . We would like to prove that the Koopmans operator is a concave operator on  $B^+$  in the sense of Krasnosels'kii and Zabriecko. Our objective is to appeal to Liang et al's [31] theorem. However, we cannot apply those results directly.

### 6.1 The Basic Idea

We begin by showing the Koopmans operator is an ordered concave operator mapping  $B^+$  to itself. However, checking the other hypotheses necessary to invoke Liang et al's [31] theorem requires a more roundabout method.

Next, we show that the Koopmans operator actually sends points in  $B^+$  to  $B_{T_W \theta}^+$  for each point in  $B^+$  with its domain restricted to  $K_{ab}$ , for any choice of  $a, b$  subject to  $0 < a < b < \infty$ . This result is obtained by showing (Lemma 17)  $B^+|K_{ab} = B_{T_W \theta}^+|K_{ab}$ . Apply Liang et al's theorem to the cone  $B^+|K_{ab}$

and notice that we obtain, by way of Krasnosels'kii and Zabriecko's theorem, uniqueness in the positive cone  $B^+|K_{ab}$ . Consider the function  $U^\infty$  found in our Theorem 6. It is defined over the choice of a function in  $B^+$ . It is sup norm upper-semicontinuous as well. We can also infer from the Corollary 9 that  $U^\infty \in B_{T_W\theta}^+$ . That is,  $U^\infty(C) \geq T_W\theta(C)$  and a strict inequality obtains whenever  $c_1 > 0$ . This is so as the *smallest* fixed point of the Koopmans operator,  $U_\infty$ , satisfies for each  $C \in \ell_\infty^+$   $U_\infty(C) = T_W U_\infty(C) \geq \theta(C) = 0$  with a strict inequality whenever  $C > 0, C \neq 0$ . Let  $U^\infty > \theta$  be shorthand for  $U^\infty(C) \geq T_W\theta(C) \geq \theta(C) = 0$  for all  $C \in \ell_\infty^+$  and a strict inequality obtains for some  $C$ . Thus  $U^\infty > \theta$  is a solution of the Koopmans equation over the choice of a function in  $B^+$ . Hence,  $U^\infty|K_{ab}$  must also solve the Koopmans equation for functions whose domains are restricted to  $K_{ab}$  as  $T_W U^\infty(C) = U^\infty(C) = W(c_1, U^\infty(SC))$  must hold for ALL  $C \in \ell_\infty^+$ , so *a fortiori*, it must also hold for all  $C \in K_{ab}$ . This observation makes use of the shift invariance property of  $K_{ab}$ . NOTE:  $U^\infty \gg \theta$  on each  $K_{ab}$  as  $W(a, U^\infty(Sa_{con})) \geq W(a, 0) > 0$  by the assumed properties of Thompson aggregators and each  $C \in K_{ab}$  satisfies  $C \geq ae \gg 0$ .

Thus if we can show there is at most one solution in  $B^+|K_{ab}$ , then  $U^\infty|K_{ab}$  is the uniquely determined solution on that domain! The choices of the parameters being essentially arbitrary implies  $U^\infty$  is the unique solution for  $B^+|\ell_\infty^+$ . This is the best we can do!!!

## 6.2 Uniqueness on Each $B^+|K_{ab}$ .

**Blanket Assumption:**  $W$  is a Thompson aggregator. It satisfies (T5), (T6), and is jointly concave in its arguments.<sup>30</sup>

Let  $B^+|K_{ab}$  be given. It is routine to verify that this set is a closed, convex cone of functions in the Banach space

$$B|K_{ab} = \{U : K_{ab} \subset \ell_\infty^+ \rightarrow \mathbb{R}_+ \text{ such that } U \text{ is } \varphi_\gamma \text{-bounded}\}.$$

Here,  $\varphi_\gamma(C) = (1 + \|C\|_\infty)^{1/\gamma}$  with  $\|C\|_\infty = \sup_t c_t$  for  $C \in K_{ab}$ .

**We also recall that  $T_W$  is a monotone operator.**

**Lemma 16**  $T_W : B^+|K_{ab} \rightarrow B^+|K_{ab}$  and  $T_W$  is an ordered-concave operator on  $B^+|K_{ab}$ .

**Proof.** That  $T_W$  is an ordered concave operator follows from the concavity of the aggregator function and convexity of the cone  $B^+|K_{ab}$ . ■

The next lemma is the crucial step in applying Liang et al's result. We seek to establish that our Koopmans operator is a  $T_W\theta$ -concave operator. We cast the cone  $B^+|K_{ab} \equiv P$  and set  $u_0 \equiv T_W\theta$ . Recall  $T_W\theta > \theta$ . Put  $B_{T_W\theta}^+|K_{ab}$  in the role of  $P_{u_0}$  as follows

$$B_{T_W\theta}^+|K_{ab} = \left\{ \begin{array}{l} U \in B^+|K_{ab} : \exists \alpha(U), \beta(U) > 0 \\ \text{such that } \alpha(U)T_W\theta \leq U \leq \beta(U)T_W\theta \end{array} \right\}.$$

<sup>30</sup>The concavity assumption imposed here is stronger than T3 and plays a crucial role in proving the Koopmans operator is both an ordered concave operator as well as a  $T_W\theta$ -concave operator.

**Lemma 17**  $T_W(B^+|K_{ab}) \subseteq B_{T_W\theta}^+|K_{ab}$ .

**Proof.** We must show that for any  $U \in B^+|K_{ab}$  there are numbers  $\alpha(U), \beta(U) > 0$  such that

$$\alpha(U) T_W\theta \leq T_W U \leq \beta(U) T_W\theta.$$

Since  $T_W$  is a monotone operator, we always have  $T_W U \geq T_W\theta$  whenever  $U \geq \theta$ . Hence, may set  $\alpha(U) = 1$ . On the other hand, since  $U$  is  $\varphi_\gamma$ -bounded, there is a number  $M^U$  such that  $U(C) \leq M^U \varphi_\gamma(C)$  for each  $C \in \ell_\infty^+$ . Also note that  $\varphi_\gamma(C) \leq \varphi_\gamma(be) = b(1+b)^{1/\gamma}$  for each  $C \in K_{ab}$ . Thus, by monotonicity of the Koopmans operator we obtain:

$$T_W U(C) \leq T_W \varphi_\gamma(C) \leq W(c_1, M^U b(1+b)^{1/\gamma})$$

for each  $C \in K_{ab}$ . Moreover,  $T_W\theta(C) = W(c_1, 0) \geq W(a, 0)$  by monotonicity of the aggregator function and the definition of  $K_{ab}$ . Since  $W(a, 0) > 0$  by Thompson property (T4), we can choose a number  $\beta(U)$  large enough such that

$$W(c_1, M^U b(1+b)^{1/\gamma}) \leq \beta(U) W(a, 0),$$

or

$$T_W U(C) \leq \beta(U) T_W\theta(C)$$

for each  $C \in K_{ab}$ . That is,

$$T_W U \leq \beta(U) T_W\theta. \tag{13}$$

This proves the inclusion  $B^+|K_{ab} \subseteq B_{T_W\theta}^+|K_{ab}$  and completes the lemma's proof. ■

Inequality (13) is readily rearranged in the form of inequality (9): for  $\mu(U) = 1/\beta(U) > 0$ :

$$\mu(U) T_W U \leq T_W\theta.$$

Combining the previous two lemmas yields:

**Corollary 18**  $T_W : B^+|K_{ab} \rightarrow B_{T_W\theta}^+|K_{ab} = B^+|K_{ab}$  is a  $T_W\theta$ -concave operator for each  $0 < a < b < \infty$ .<sup>31</sup>

**Proof.** The two previous lemmas imply the hypotheses of Liang et al's Theorem are satisfied (specifically, (9) obtains). Hence  $T_W$  is a  $T_W\theta$ -concave operator acting on each cone  $B^+|K_{ab}$ . ■

We invoke the *Krasnosel'skiĭ and Zabreĭko Theorem* on each cone  $B^+|K_{ab}$ . This yields the existence of **at most one non-zero fixed point in  $B^+|K_{ab}$** .

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<sup>31</sup>Liang et al's approach gives us a way to verify KZ condition (9) once we have verified their first condition. The difficulty in checking the second condition is the need to construct the constant  $\eta$  to only vary with the parameter  $t$  and the function  $U$ . Thompson property T3 can, by itself provide a pointwise bound of the type required, but the  $\eta$  depends on  $t, U$ , and  $C$ . The theorem is not valid unless  $\eta$  is independent of  $C$ . The KZ theorem is applied to each cone  $B^+|K_{ab}$  with given parameters  $a$  and  $b$ .

**Proposition 19** *For each given  $0 < a < b < \infty$ , the Koopmans operator equation,  $T_W U = U$ , has at most one non-zero solution in the cone  $B^+|K_{ab}$ .*

Suppose that there is a non-zero fixed point on  $B^+|K_{ab}$ . Denote that non-zero fixed point by  $U^{ab}$ . That is, let  $U^{ab} = T_W U^{ab}$ . Theorem 6 implies there is always at least one non-zero solution in the cone  $B^+$ . In fact, that solution also lies in a particular order interval:

$$\langle \theta, U^T \rangle \subset B^+, \text{ where } U^T(C) = \varphi_\gamma(C)$$

since the growth factor is  $\alpha = 1$  as in Section 4's reconstruction of Marinacci and Montrucchio's existence theory.

Moreover,  $U^\infty$  is a sup norm upper-semicontinuous real-valued function on  $\ell_\infty^+$ . Denote the restriction of  $U^\infty$  to the subset  $K_{ab} \subset \ell_\infty^+$  as  $U_{ab}^\infty$ . That is,  $U^\infty|K_{ab} \equiv U_{ab}^\infty$ . Clearly we must also have  $U^{ab} = U_{ab}^\infty$  since  $U^{ab}$  is the unique non-zero solution to the Koopmans equation on the domain  $B^+|K_{ab}$  by Proposition 19. But clearly, we have  $U_{ab}^\infty \in B^+|K_{ab}$  and  $U_{ab}^\infty \neq \theta$ . Hence,

$$T_W U_{ab}^\infty = U_{ab}^\infty$$

for each  $0 < a < b < \infty$ . Therefore, we may conclude that  $U^\infty$  is the unique solution among all the  $\varphi$ -bounded nonnegative real-valued functions defined on  $\ell_\infty^+$ . This also implies that  $U^\infty = U_\infty = \sup_N (T_W^N \theta)$ , where  $T_W^N$  is the  $N^{\text{th}}$  - iterate of the Koopmans operator with initial seed function  $\theta$ . The fact that  $U_\infty$  is a sup norm lower-semicontinuous implies that  $U^\infty$  is a sup norm continuous function on  $\ell_\infty^+$ .

We sum up our findings:

**Theorem 20** *Let  $W$  be a Thompson aggregator. Suppose  $W$  is  $\gamma$ -subhomogeneous of degree  $\gamma$ , satisfies the MM - Limit Condition, and is jointly continuous and concave in both of its arguments. Then, there is a unique norm-continuous utility function  $U^\infty \in \langle \theta, U^T \rangle$ , and defined on  $\ell_\infty^+$ , that solves the Koopmans equation in the sense that*

$$T_W U^\infty(C) = U^\infty(C)$$

for each  $C \in \ell_\infty^+$ . In particular,  $U^\infty(C) = W(c_1, U^\infty(SC))$  holds for each  $C \in \ell_\infty^+$ .

## 7 Concluding Comments

Our existence and uniqueness theorems are not yet in a form ready for applications in problems such as the Ramsey optimal growth model. The constraint sets defining the feasible consumption plans arising in well-behaved growth models are not typically  $\|\bullet\|_\alpha$  - norm compact even though they are  $\|\bullet\|_\alpha$  - norm bounded in many cases. Indeed, the order intervals in  $\ell_\infty^+(\alpha)$  are not compact in their norm topology. They are however, compact in the product topology.

Feasible sets are typically closed subsets of an order interval and are therefore compact in the product topology. These feasible sets are also convex in standard growth models. In order to apply standard existence theorems for an optimum built from the classical Weierstrass Theorem we need to establish that  $U^\infty$  is also upper semi-continuous in the product topology on  $\ell_\infty^+(\alpha)$ . However, the initial seed function  $U^T$  for the iterative construction of  $U^\infty$  is not a product continuous function. The weight function's  $\|\bullet\|_\alpha$  - norm continuity appearing in  $U^T$  depends directly on the continuity of the  $\|\bullet\|_\alpha$  - norm on  $\ell_\infty^+(\alpha)$ . To see the problem, just let  $\alpha = 1$  and set  $\ell_\infty^+ = \ell_\infty^+(1)$ . Identify  $\|\bullet\|$  and  $\|\bullet\|_1$ . Suppose the corresponding  $\|\bullet\|$  - norm is a continuous function in the topology of coordinatewise convergence. Let  $C^1 = \{1, 0, 0, 0, \dots\}$ ,  $C^2 = \{0, 1, 0, 0, \dots\}$ , and so on. Each sequence belongs to  $\ell_\infty^+$  and  $\|C^N\| = 1$  for each  $N$ . But  $\{C^N\} \rightarrow 0$  in the topology of coordinatewise convergence. Hence, if  $\|\bullet\|$  is a continuous function in this topology,  $\lim_N \|C^N\| = 0$  as well. This contradicts the property  $\|C^N\| = 1$  for each  $N$ . Hence,  $\|\bullet\|$  is not continuous in the topology of coordinatewise convergence.<sup>32</sup>

The zero function  $\theta$  is continuous in the topology of coordinatewise convergence and  $U_\infty$  is lower-semicontinuous in this topology as well. IF there is a unique solution to the Koopmans equation so that  $U_\infty = U^\infty$ , then we can infer that  $U^\infty$  is also lower-semicontinuous in the topology of coordinatewise convergence. The functions  $U^\infty$  and  $U_\infty$  constructed from Tarski-Kantorovich are independent of the topology chosen for  $\ell_\infty^+(\alpha)$  — they depend only the order structure embedded in  $\langle \theta, U^T \rangle$  via the order continuity property of the Koopmans operator. However, the upper semicontinuity of  $U^\infty$  does depend on the continuity of the function  $U^T$  on  $\ell_\infty^+(\alpha)$ . IF we cannot input a continuous initial seed,  $U^T$ , we cannot infer from iteration of the Koopmans operator that  $U^\infty$  enjoys any continuity property! Weakening the topology from the  $\|\bullet\|$  - norm topology risks losing the continuity of this initial seed when we apply Tarski-Kantorovich.<sup>33</sup> The question then becomes how far can the topology on  $\ell_\infty^+(1)$  be weakened to provide compact feasible sets in standard growth models AND an upper-semicontinuous  $U^\infty$ . In such a case, if there is a unique solution to the Koopmans operator we can conclude it is also continuous in that alternative topology. Martins-Da-Rocha and Vailakis [36] prove uniqueness theorems on a similar domain as ours, but employing weaker topologies that would be consistent with proving optimal programs exist. Their results turn on the local contraction mapping theorems originating in Rincón-Zapatero and Rodriguez-Palmero ([41], [42]), Marinacci and Montrucchio [37], and Martins-da-Rocha and Vailakis ([35], [36]).

Our uniqueness theorem restricts the commodity space in order to rule out sequences of consumption that are eventually zero, or even converge to zero. These restrictions are not stronger than similar ones in Martins-da-Rocha and

<sup>32</sup>See Majumdar [33] for a closely related discussion about why compactness of feasible sets fails in the one-sector growth model when the commodity space  $\ell_\infty$  is given its sup norm topology.

<sup>33</sup>The sup norm is weakly lower semicontinuous on  $\ell_\infty^+(\alpha)$ . See Aliprantis and Border ([1], p. 235).

Vailakis [35]. We revise the Marinacci and Montrucchio [37] properties defining Thompson aggregators. Our modified conditions include joint concavity and joint continuity of the aggregator  $W$  on its domain. The joint concavity assumption is stronger than their (T3). We strengthen (T3) in order to implement our concave operator based uniqueness theory. Our uniqueness theory differs from the existing approaches in the literature. First, our utility functions domains differ. Second, our concave operator technique is an alternative to the contraction operator theorems with Thompson metrics, or the 0-local contractions introduced by Rincón-Zapatero and Rodríguez-Palmero ([41], [42]), Marinacci and Montrucchio [37], and Martins-da-Rocha and Vailakis ([35], [36]). It is worth noting the uniqueness contribution by Marinacci and Montrucchio [37] accommodates some endogenous growth models, whereas we are not able to do so.

The  $u_0$ -concave operator theory required one important economic restriction. The underlying commodity space could NOT admit sustainable growth or balanced growth paths. It would be of some interest to work out a uniqueness theory for the same principal ideal commodity spaces supporting the general existence theory for solutions to the Koopmans equation. Such a generalization would supply a uniqueness theorem that accommodates endogenous growth as well.

## 8 Appendix: Koopmans Equivalent Aggregators

Consider standard demand theory with two goods,  $x$  and  $y$ . Identify the aggregator function as the utility function defined over those two goods. Clearly an increasing transformation of this utility function defines an ordinally equivalent utility function. No behavioral implications of maximizing behavior are altered by this change of utility units. This ordinal equivalence property of utility changes in the context of recursive utility even though the resulting transformed aggregator has the same CES parameter value as before! Consider

$$V(x, y) = [(1 - \beta)x^\rho + \beta y^\rho]^{1/\rho}, \text{ for } 0 < \rho < 1. \quad (14)$$

Its elasticity of substitution equal to  $1/(1 - \rho)$  — the same as the aggregator  $W$  in (3) provided  $\rho \neq 1$ . It is well-known that  $V$  converges to the Cobb-Douglas function when  $\rho \rightarrow 0$

Note that  $V$  is  $\rho$ -homogeneous. We refer to the two parameter class of aggregators defined in (14) as the  $V$  – **class of CES aggregators** in contrast to the  $W$  – **class of CES aggregators** specified in (3). It turns out that the aggregators  $W$  and  $V$  so-defined represent different preferences in the recursive utility world with infinite horizons! That is, setting  $V^*(x, y) = [V(x, y)]^\rho = W(x, y)$  produces an aggregator that turns out to admit different ordinal properties of utility than the specification of  $V(x, y)$ ! These aggregators share a common elasticity of substitution in classical demand theory setting, but their ordinally equivalent versions do not share this common property in the recursive utility aggregator framework!

Two arbitrary aggregators  $W$  and  $W^*$  are said to be **equivalent in Koopmans’ [24] sense**, or simply **Koopmans equivalent**, if their corresponding utility functions represent the same preference ordering over the underlying commodity space. This equivalence amounts to saying the aggregator ranks alternatives in a manner that is ordinal — the ranking over  $(x, y)$  combinations is invariant under increasing transformations of the aggregator in the following manner: there is a strictly increasing function  $\Phi$  such that

$$W^*(x, y) = \Phi [W(x, \Phi^{-1}(y))] . \quad (\text{E})$$

Koopmans’ notion of aggregator equivalence reflects the fact that the utility function is on the left-hand and right-hand sides of his equation:  $U(C) = W(c_1, U(SC))$ . This self-referential feature of recursive utility implies that an increasing transformation,  $\Phi$ , of a utility function, as reflected through the aggregator via Koopmans equation, must also be “undone” to provide a consistent ranking and this is how the inverse function  $\Phi^{-1}$  ends up in the aggregator’s second argument. Indifference curves of the aggregator in the coordinate system  $(x, y)$  change to a new coordinate system when  $\Phi$  is applied to both sides of Koopmans equation as  $\Phi$  must also act on the  $y$  – *coordinate*. Thus, the increasing transformation “stretches” the  $y$  – *axis* to have units  $\Phi(y)$  and this change of units must be undone to write the transformed aggregator,  $W^*$ , as a function of  $x$  and the original variable  $y$ . Doing so can sometimes create surprising connections between aggregator specifications.

Consider the CES aggregators  $V$  as defined in (14). Let  $\Phi(y) = y^\rho$  with  $\Phi^{-1}(y) = y^{1/\rho}$  for  $0 < \rho < 1$ . Using this increasing transformation we find that the corresponding Koopmans equivalent aggregator is

$$V^*(x, y) = (1 - \beta)x^\rho + \beta y . \quad (15)$$

Clearly  $V^*$  is linear in  $y$  — the parameter  $\beta$  corresponds to a discount factor and this aggregator specifies a familiar time additive separable utility function since it is a Blackwell aggregator!<sup>34</sup> Thus, a monotone transformation of  $V$  (in (14) yielding  $W$  (in 3) does NOT produce a Koopmans equivalent aggregator since  $V$  is nonlinear in  $y$  and the correct Koopmans equivalent aggregator for this transformation,  $V^*$ , is linear in  $y$ . This example shows that elasticity of substitution can vary among Koopmans equivalent aggregators. Moreover, the same elasticity of substitution between current consumption and future utility can be shared by aggregators which fail to be Koopmans equivalent. The elasticity of substitution between current consumption and future utility is **not** an ordinal property of the aggregator or the infinite horizon preference relation!

**Koopmans Equivalent Aggregators and the MM Limit Condition** An interesting feature of the CES aggregator family is that its members do not automatically satisfy the MM Limit Condition imposed by (T6). Again, let

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<sup>34</sup>Of course, this specification of the aggregator also satisfies the basic Thompson criteria.

$L = \lim_{y \rightarrow \infty} V(1, y)/y$ . Suppose  $V$  is a  $V$ -class CES aggregator with  $0 < \rho < 1$ . Factor  $y^\rho$  out in the numerator and observe:

$$L = \lim_{y \rightarrow \infty} \frac{((1 - \beta) + \beta y^\rho)^{1/\rho}}{y} = \lim_{y \rightarrow \infty} \left( ((1 - \beta) y^{-\rho} + \beta)^{1/\rho} \right) = \beta^{1/\rho}$$

since  $0 < \rho < 1$  implies  $y^{-\rho} \rightarrow 0$  as  $y \rightarrow \infty$ . Hence, the MM Limit Condition obtains if and only if the inequality

$$\beta^{1/\rho} < \alpha^{-1/\gamma} = \alpha^{-1}$$

since  $\gamma = 1$ . That is, for (1) to obtain, the condition  $\alpha\beta^{1/\rho} < 1$  must hold. This condition is a **joint condition on preferences via  $\beta$  and  $\rho$ , as well as the maximum consumption growth factor,  $\alpha$** . Notice this is identical to the same restriction for the Koopmans equivalent Blackwell aggregator  $W^*$  required to derive a unique solution to the Koopmans equation!<sup>35</sup> Property (T6) can impose a real constraint on the deep preference and technology parameters underlying a given economic (growth) model when the CES  $V$ -class Thompson form is postulated and the underlying utility function is sought as the corresponding Koopmans' equation solution.

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<sup>35</sup>See Becker and Boyd ([9], pp. 101-103) for details on this Blackwell case.



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